**District Olympiad 2019**

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| – | Grade 9 |
| 1 | Let $n \in \mathbb{N}, n \geq 2$ and the positive real numbers $a_1, a_2, , a_n$ and $b_1, b_2, , b_n$ such that $a_1+a_2++a_n = b_1+b_2+ +b_n = S$. **a)** Prove that $\sum_{k=1}^{n} \frac{a_k^2}{a_k+b_k} \geq \frac{S}{2}$. **b)** Prove that $\sum_{k=1}^{n} \frac{a_k^2}{a_k+b_k} = \sum_{k=1}^{n} \frac{b_k^2}{a_k+b_k}$. |
| 2 | Let $H$ be the orthocenter of the acute triangle $ABC$. In the plane of the triangle $ABC$ we consider a point $X$ such that the triangle $XAH$ is right and isosceles, having the hypotenuse $AH$, and $B$ and $X$ are on each part of the line $AH$. Prove that $\overrightarrow{XA}+\overrightarrow{XC}+\overrightarrow{XH}=\overrightarrow{XB}$ if and only if $\angle BAC = 45^\circ$. |
| 3 | Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $$2(a_1+a_2+ +a_n) = na_{n+1}\ \forall\, n \geq 1.$$ **a)** Prove that the given sequence is an arithmetic progression. **b)** If $\lfloor a_1 \rfloor + \lfloor a_2 \rfloor + + \lfloor a_n \rfloor = \lfloor a_1+a_2+ +a_n \rfloor\ \forall\, n \in \mathbb{N}$, prove that every term of the sequence is an integer. |
| 4 | Find all positive integers $p$ for which there exists a positive integer $n$ such that $p^n+3^n \mid p^{n+1}+3^{n+1}$. |
| – | Grade 10 |
| 1 | Find the functions $f:\mathbb{R}\to(0,\infty)$ which satisfy $$2^{-x-y} \leq \frac{f(x)f(y)}{(x^2+1)(y^2+1)} \leq \frac{f(x+y)}{(x+y)^2+1},$$ for all $x, y \in \mathbb{R}$. |
| 2 | Let $n \in \mathbb{N}, n \geq 3$. $a)$ Prove that there exist $z_1, z_2, , z_n \in \mathbb{C}$ such that $$\frac{z_1}{z_2}+\frac{z_2}{z_3}+ +\frac{z_{n-1}}{z_n}+\frac{z_n}{z_1} = ni.$$ $b)$ Which are the values of $n$ for which there exist the complex numbers $z_1, z_2, , z_n$, of the same modulus, such that $$\frac{z_1}{z_2}+\frac{z_2}{z_3}+ +\frac{z_{n-1}}{z_n}+\frac{z_n}{z_1} = ni?$$ |

**3** Let $a, b, c$ be distinct complex numbers with $|a| = |b| = |c| = 1$. If $|a+b-c|^2 + |b+c-a|^2 + |c+a-b|^2 = 12$, prove that the points of affixes $a, b, c$ are the vertices of an equilateral triangle.

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**4** Find the smallest positive real number $\lambda$ such that for every numbers $a_1, a_2, a_3 \in \left[0, \frac{1}{2}\right]$ and $b_1, b_2, b_3 \in (0, \infty)$ with $\sum\limits_{i=1}^{3} a_i = \sum\limits_{i=1}^{3} b_i = 1$, we have

$$b_1 b_2 b_3 \le \lambda (a_1 b_1 + a_2 b_2 + a_3 b_3).$$

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**–** Grade 11

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**1** Let $(a_n)_{n\ge 1}$ be a sequence of positive real numbers such that the sequence $(a_{n+1} - a_n)_{n\ge 1}$ is convergent to a non-zero real number. Evaluate the limit

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^n.$$

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**2** Let $n \in \mathbb{N}, n \ge 2$, and $A, B \in \mathcal{M}_n(\mathbb{R})$. Prove that there exists a complex number $z$, such that $|z| = 1$ and

$$\Re\left(\det(A + zB)\right) \ge \det(A) + \det(B),$$

where $\Re(w)$ is the real part of the complex number $w$.

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**3** Let $n$ be an odd natural number and $A, B \in \mathcal{M}_n(\mathbb{C})$ be two matrices such that $(A-B)^2 = O_n$. Prove that $\det(AB - BA) = 0$.

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**4** Let $f : [0, \infty) \to [0, \infty)$ be a continuous function with $f(0) > 0$ and having the property

$$x - y < f(y) - f(x) \le 0 \; \forall \; 0 \le x < y.$$

Prove that: $a)$ There exists a unique $\alpha \in (0, \infty)$ such that $(f \circ f)(\alpha) = \alpha$. $b)$ The sequence $(x_n)_{n\ge 1}$, defined by $x_1 \ge 0$ and $x_{n+1} = f(x_n) \; \forall \; n \in \mathbb{N}$ is convergent.

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**–** Grade 12

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**1** Let $n$ be a positive integer and $G$ be a finite group of order $n$. A function $f : G \to G$ has the $(P)$ property if $f(xyz) = f(x)f(y)f(z) \; \forall \; x, y, z \in G$. **(a)** If $n$ is odd, prove that every function having the $(P)$ property is an endomorphism. **(b)** If $n$ is even, is the conclusion from **(a)** still true?

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**2** Let $n$ be a positive integer and $f : [0, 1] \to \mathbb{R}$ be an integrable function. Prove that there exists a point $c \in \left[0, 1 - \frac{1}{n}\right]$, such that

$$\int_c^{c+\frac{1}{n}} f(x)\mathrm{d}x = 0 \text{ or } \int_0^c f(x)\mathrm{d}x = \int_{c+\frac{1}{n}}^1 f(x)\mathrm{d}x.$$

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**3** Let $G$ be a finite group and let $x_1, , x_n$ be an enumeration of its elements. We consider the matrix $(a_{ij})_{1\le i,j\le n}$, where $a_{ij} = 0$ if $x_i x_j^{-1} = x_j x_i^{-1}$, and $a_{ij} = 1$ otherwise. Find the parity of the integer $\det(a_{ij})$.

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**4** Let $a$ be a real number, $a > 1$. Find the real numbers $b \ge 1$ such that

$$\lim_{x\to\infty} \int_0^x (1+t^a)^{-b}\mathrm{d}t = 1.$$

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