Turkey Team Selection Test 2019

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Day 1 23 March 2019

1 In each one of the given 2019 boxes, there are 2019 stones numbered as $1, 2, ..., 2019$ with total mass of 1 kilogram. In all situations satisfying these conditions, if one can pick stones from different boxes with different numbers, with total mass of at least 1 kilogram, in k different ways, what is the maximal of k?

2 $(a_n)_{n=1}^{\infty}$ is an integer sequence, $a_1 = 1, a_2 = 2$ and for $n \geq 1, a_{n+2} = a_{n+1}^2 + (n+2)a_{n+1} - a_n^2 - na_n$.
$a)$ Prove that the set of primes that divides at least one term of the sequence can not be finite.
$b)$ Find 3 different prime numbers that do not divide any terms of this sequence.

3 In a triangle ABC, $AB > AC$. The foot of the altitude from A to BC is D, the intersection of bisector of B and AD is K, the foot of the altitude from B to CK is M and let BM and AK intersect at point N. The line through N parallel to DM intersects AC at T. Prove that BM is the bisector of angle $\widehat{TBC}$.

Day 2 24 March 2019

4 For an integer n with b digits, let a *subdivisor* of n be a positive number which divides a number obtained by removing the r leftmost digits and the l rightmost digits of n for nonnegative integers r, l with $r + l < b$ (For example, the subdivisors of 143 are 1, 2, 3, 4, 7, 11, 13, 14, 43, and 143). For an integer d, let A_d be the set of numbers that don't have d as a subdivisor. Find all d, such that A_d is finite.

5 $P(x)$ is a nonconstant polynomial with real coefficients and its all roots are real numbers. If there exist a $Q(x)$ polynomial with real coefficients that holds the equality for all x real numbers
$(P(x))^2 = P(Q(x))$,
then prove that all the roots of $P(x)$ are same.

6 k is a positive integer, $R_n = -k, -(k-1), ..., -1, 1, ..., k-1, k$ for $n = 2k$ $R_n = -k, -(k-1), ..., -1, 0, 1, ..., k$
for $n = 2k+1$.
A mechanism consists of some marbles and white/red ropes that connects some marble pairs. If each one of the marbles are written on some numbers from R_n with the property that any two connected marbles have different numbers on them, we call it *nice labeling*. If each one of the marbles are written on some numbers from R_n with the properties that any two connected marbles with a white rope have different numbers on them and any two connected marbles with a red rope have two numbers with sum not equal to 0, we call it *precise labeling*.

$n \geq 3$, if every mechanism that is labeled *nicely* with R_n, could be labeled *precisely* with R_m, what is the minimal value of m?

Day 3 25 March 2019

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7 In a triangle ABC with $\angle ACB = 90^\circ$ D is the foot of the altitude of C. Let E and F be the reflections of D with respect to AC and BC. Let O_1 and O_2 be the circumcenters of $\triangle ECB$ and $\triangle FCA$. Show that:

$$2O_1O_2 = AB$$

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8 Let $p > 2$ be a prime number, $m > 1$ and n be positive integers such that $\frac{m^{pn}-1}{m^n-1}$ is a prime number. Show that:

$$pn \mid (p-1)^n + 1$$

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9 Let x, y, z be real numbers such that $y \geq 2z \geq 4x$ and

$$2(x^3 + y^3 + z^3) + 15(xy^2 + yz^2 + zx^2) \geq 16(x^2y + y^2z + z^2x) + 2xyz.$$

Prove that: $4x + y \geq 4z$
