Final Round - Korea 2019

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Day 1 March 23rd

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1 There are n cards such that for each $i = 1, 2, \cdots n$, there are exactly one card labeled i. Initially the cards are piled with increasing order from top to bottom. There are two operations:

- A : One can take the top card of the pile and move it to the bottom;

- B : One can remove the top card from the pile.

The operation $ABBABBABBABB\cdots$ is repeated until only one card gets left. Let $L(n)$ be the labeled number on the final pile. Find all integers k such that $L(3k) = k$.

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2 For a rectangle $ABCD$ which is not a square, there is O such that O is on the perpendicular bisector of BD and O is in the interior of $\triangle BCD$. Denote by E and F the second intersections of the circle centered at O passing through B, D and AB, AD. BF and DE meets at G, and X, Y, Z are the foots of the perpendiculars from G to AB, BD, DA. L, M, N are the foots of the perpendiculars from O to CD, BD, BC. XY and ML meets at P, YZ and MN meets at Q. Prove that BP and DQ are parallel.

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3 Prove that there exist infinitely many positive integers k such that the sequence $\{x_n\}$ satisfying

$$x_1 = 1, x_2 = k+2, x_{n+2} - (k+1)x_{n+1} + x_n = 0 (n \geq 0)$$

does not contain any prime number.

Day 2 March 24th

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4 Let triangle ABC be an acute scalene triangle with orthocenter H. The foot of perpendicular from A to BC is O, and denote K, L by the midpoints of AB, AC, respectively. For a point $D(\neq O, B, C)$ on segment BC, let E, F be the orthocenters of triangles ABD, ACD, respectively, and denote M, N by the midpoints of DE, DF. The perpendicular line from M to KH cuts the perpendicular line from N to LH at P. If Q is the midpoint of EF, and S is the orthocenter of triangle HPQ, then prove that as D varies on BC, the ratio $\frac{OS}{OH}, \frac{OQ}{OP}$ remains constant.

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5 Find all pairs (p, q) such that the equation

$$x^4 + 2px^2 + qx + p^2 - 36 = 0$$

has exactly 4 integer roots(counting multiplicity).

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6 A sequence $\{x_n\} = x_0, x_1, x_2, \cdots$ satisfies $x_0 = a(1 \le a \le 2019, a \in \mathbb{R})$, and

$$x_{n+1} = \begin{cases} 1 + 1009x_n & (x_n \le 2) \\ 2021 - x_n & (2 < x_n \le 1010) \\ 3031 - 2x_n & (1010 < x \le 1011) \\ 2020 - x_n & (1011 < x_n) \end{cases}$$

for each non-negative integer n. If there exist some integer $k > 1$ such that $x_k = a$, call such minimum k a *fundamental period* of $\{x_n\}$. Find all integers which can be a fundamental period of some seqeunce; and for such minimal odd period $k(> 1)$, find all values of $x_0 = a$ such that the fundamental period of $\{x_n\}$ equals k.
