2019 Pan-African Mathematics Olympiad

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1 Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers defined as follows:

- $a_0 = 3$, $a_1 = 2$, and $a_2 = 12$; and
- $2a_{n+3} - a_{n+2} - 8a_{n+1} + 4a_n = 0$ for $n \geq 0$.

Show that a_n is always a strictly positive integer.

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2 Let k be a positive integer. Consider k not necessarily distinct prime numbers such that their product is ten times their sum. What are these primes and what is the value of k?

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3 Let ABC be a triangle, and D, E, F points on the segments BC, CA, and AB respectively such that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB}.$$

Show that if the centres of the circumscribed circles of the triangles DEF and ABC coincide, then ABC is an equilateral triangle.

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4 The tangents to the circumcircle of $\triangle ABC$ at B and C meet at D. The circumcircle of $\triangle BCD$ meets sides AC and AB again at E and F respectively. Let O be the circumcentre of $\triangle ABC$. Show that AO is perpendicular to EF.

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5 A square is divided into N^2 equal smaller non-overlapping squares, where $N \geq 3$. We are given a broken line which passes through the centres of all the smaller squares (such a broken line may intersect itself).

- Show that it is possible to find a broken line composed of 4 segments for $N = 3$.
- Find the minimum number of segments in this broken line for arbitrary N.

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6 Find the 2019th strictly positive integer n such that $\binom{2n}{n}$ is not divisible by 5.
