**EGMO 2019**

www.artofproblemsolving.com/community/c858696

by Tintarn, juckter, rmtf1111

| | | |
|---|---|---|
| – | Day 1 | |

**1** Find all triples $(a, b, c)$ of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

**2** Let $n$ be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each $n$, determine the largest number of dominoes that can be placed in this way.
(A domino is a tile of size $2 \times 1$ or $1 \times 2$. Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be adjacent if they are different and share a common side.)

**3** Let $ABC$ be a triangle such that $\angle CAB > \angle ABC$, and let $I$ be its incentre. Let $D$ be the point on segment $BC$ such that $\angle CAD = \angle ABC$. Let $\omega$ be the circle tangent to $AC$ at $A$ and passing through $I$. Let $X$ be the second point of intersection of $\omega$ and the circumcircle of $ABC$. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line $BC$.

**–** Day 2

**4** Let $ABC$ be a triangle with incentre $I$. The circle through $B$ tangent to $AI$ at $I$ meets side $AB$ again at $P$. The circle through $C$ tangent to $AI$ at $I$ meets side $AC$ again at $Q$. Prove that $PQ$ is tangent to the incircle of $ABC$.

**5** Let $n \geq 2$ be an integer, and let $a_1, a_2, \cdots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \cdots, b_n$ satisfying the following three conditions:

(A) $a_i \leq b_i$ for $i = 1, 2, \cdots, n;$

(B) the remainders of $b_1, b_2, \cdots, b_n$ on division by $n$ are pairwise different; and

(C) $b_1 + b_2 + \cdots b_n \leq n\left(\frac{n-1}{2} + \left\lfloor \frac{a_1+a_2+\cdots a_n}{n} \right\rfloor\right)$

(Here, $\lfloor x \rfloor$ denotes the integer part of real number $x$, that is, the largest integer that does not exceed $x$.)

**6** On a circle, Alina draws $2019$ chords, the endpoints of which are all different. A point is considered *marked* if it is either

(i) one of the $4038$ endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the $4038$ points meeting criterion (i), Alina labels $2019$ points with a $0$ and the other $2019$ points with a $1$. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).
Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with $k$ marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.
Alina finds that the $N+1$ yellow labels take each value $0, 1, ..., N$ exactly once. Show that at least one blue label is a multiple of $3$.
(A chord is a line segment joining two different points on a circle.)