Canada National Olympiad 2019

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1 Points A, B, C are on a plane such that $AB = BC = CA = 6$. At any step, you may choose any three existing points and draw that triangle's circumcentre. Prove that you can draw a point such that its distance from an previously drawn point is: (a) greater than 7 (b) greater than 2019

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2 Let a, b be positive integers such that $a + b^3$ is divisible by $a^2 + 3ab + 3b^2 - 1$. Prove that $a^2 + 3ab + 3b^2 - 1$ is divisible by the cube of an integer greater than 1.

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3 You have a $2m$ by $2n$ grid of squares coloured in the same way as a standard checkerboard. Find the total number of ways to place mn counters on white squares so that each square contains at most one counter and no two counters are in diagonally adjacent white squares.

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4 Prove that for $n > 1$ and real numbers $a_0, a_1, \ldots, a_n, k$ with $a_1 = a_{n-1} = 0$,

$$|a_0| - |a_n| \leq \sum_{i=0}^{n-2} |a_i - ka_{i+1} - a_{i+2}|.$$

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5 A 2-player game is played on $n \geq 3$ points, where no 3 points are collinear. Each move consists of selecting 2 of the points and drawing a new line segment connecting them. The first player to draw a line segment that creates an odd cycle loses. (An odd cycle must have all its vertices among the n points from the start, so the vertices of the cycle cannot be the intersections of the lines drawn.) Find all n such that the player to move first wins.
