Iran Team Selection Test 2019

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by Dadgarnia, Hamper.r, M.Sharifi

Test 1 Day 1

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1 A table consisting of 5 columns and 32 rows, which are filled with zero and one numbers, are "varied", if no two lines are filled in the same way.

On the exterior of a cylinder, a table with 32 rows and 16 columns is constructed. Is it possible to fill the numbers cells of the table with numbers zero and one, such that any five consecutive columns, table 32×5 created by these columns, is a varied one?

Proposed by Morteza Saghafian

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2 $a, a_1, a_2, \dots, a_n$ are natural numbers. We know that for any natural number k which $ak+1$ is square, at least one of $a_1k+1, \dots, a_nk+1$ is also square.
Prove a is one of $a_1, \dots, a_n$
Proposed by Mohsen Jamali

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3 Point P lies inside of parallelogram $ABCD$. Perpendicular lines to PA, PB, PC and PD through A, B, C and D construct convex quadrilateral $XYZT$. Prove that $S_{XYZT} \geq 2S_{ABCD}$.

Proposed by Siamak Ahmadpour

Test1 Day 2

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4 Consider triangle ABC with orthocenter H. Let points M and N be the midpoints of segments BC and AH. Point D lies on line MH so that $AD \parallel BC$ and point K lies on line AH so that $DNMK$ is cyclic. Points E and F lie on lines AC and AB such that $\angle EHM = \angle C$ and $\angle FHM = \angle B$. Prove that points D, E, F and K lie on a circle.

Proposed by Alireza Dadgarnia

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5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f\left(f(x)^2 - y^2\right)^2 + f(2xy)^2 = f\left(x^2 + y^2\right)^2$$

Proposed by Ali Behrouz - Mojtaba Zare Bidaki

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6 $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are two sequences of positive integers that $a_i, b_i \in \{0, 1, 2, \cdots, 9\}$. There is an integer number M such that $a_n, b_n \neq 0$ for all $n \geq M$ and for each $n \geq 0$

$$(\overline{a_n \cdots a_1 a_0})^2 + 999 \mid (\overline{b_n \cdots b_1 b_0})^2 + 999$$

prove that $a_n = b_n$ for $n \geq 0$.

(Note that $(\overline{x_n x_{n-1} \dots x_0}) = 10^n \times x_n + \dots + 10 \times x_1 + x_0$.)

Proposed by Yahya Motevassel

Test 2 Day 1

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1 S is a subset of Natural numbers which has infinite members.

$$S = \{x^y + y^x : x, y \in S, x \neq y\}$$

Prove the set of prime divisors of S has also infinite members
Proposed by Yahya Motevassel

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2 In a triangle ABC, $\angle A$ is 60°. On sides AB and AC we make two equilateral triangles (outside the triangle ABC) ABK and ACL. CK and AB intersect at S , AC and BL intersect at R , BL and CK intersect at T. Prove the radical centre of circumcircle of triangles BSK, CLR and BTC is on the median of vertex A in triangle ABC.
Proposed by Ali Zamani

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3 Numbers m and n are given positive integers. There are mn people in a party, standing in the shape of an $m \times n$ grid. Some of these people are police officers and the rest are the guests. Some of the guests may be criminals. The goal is to determine whether there is a criminal between the guests or not.

Two people are considered *adjacent* if they have a common side. Any police officer can see their adjacent people and for every one of them, know that they're criminal or not. On the other hand, any criminal will threaten exactly one of their adjacent people (which is likely an officer!) to murder. A threatened officer will be too scared, that they deny the existence of any criminal between their adjacent people.

Find the least possible number of officers such that they can take position in the party, in a way that the goal is achievable. (Note that the number of criminals is unknown and it is possible to have zero criminals.)

Proposed by Abolfazl Asadi

Test 2 Day 2

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4 Let $1 < t < 2$ be a real number. Prove that for all sufficiently large positive integers like d, there is a monic polynomial $P(x)$ of degree d, such that all of its coefficients are either $+1$ or -1 and

$$|P(t) - 2019| < 1.$$

Proposed by Navid Safaei

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5 Let P be a simple polygon completely in C, a circle with radius 1, such that P does not pass through the center of C. The perimeter of P is 36. Prove that there is a radius of C that intersects P at least 6 times, or there is a circle which is concentric with C and have at least 6 common points with P.

Proposed by Seyed Reza Hosseini

6

For any positive integer n, define the subset S_n of natural numbers as follow

$$S_n = \left\{x^2 + ny^2 : x, y \in \mathbb{Z}\right\}.$$

Find all positive integers n such that there exists an element of S_n which doesn't belong to any of the sets $S_1, S_2, \dots, S_{n-1}$.

Proposed by Yahya Motevassel

Test 3 Day 1

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1 Find all polynomials $P(x, y)$ with real coefficients such that for all real numbers x, y and z:

$$P(x, 2yz) + P(y, 2zx) + P(z, 2xy) = P(x + y + z, xy + yz + zx).$$

Proposed by Sina Saleh

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2 Hesam chose 10 distinct positive integers and he gave all pairwise $\gcd$'s and pairwise lcm's (a total of 90 numbers) to Masoud. Can Masoud always find the first 10 numbers, just by knowing these 90 numbers?

Proposed by Morteza Saghafian

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3 In triangle ABC, M, N and P are midpoints of sides BC, CA and AB. Point K lies on segment NP so that AK bisects $\angle BKC$. Lines MN, BK intersects at E and lines MP, CK intersects at F. Suppose that H be the foot of perpendicular line from A to BC and L the second intersection of circumcircle of triangles AKH, HEF. Prove that MK, EF and HL are concurrent.

Proposed by Alireza Dadgarnia

Test 3 Day 2

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4 Given an acute-angled triangle ABC with orthocenter H. Reflection of nine-point circle about AH intersects circumcircle at points X and Y. Prove that AH is the external bisector of $\angle XHY$.

Proposed by Mohammad Javad Shabani

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5 A sub-graph of a complete graph with n vertices is chosen such that the number of its edges is a multiple of 3 and degree of each vertex is an even number. Prove that we can assign a weight to each triangle of the graph such that for each edge of the chosen sub-graph, the sum of the weight of the triangles that contain that edge equals one, and for each edge that is not in the sub-graph, this sum equals zero.

Proposed by Morteza Saghafian

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6 x, y and z are real numbers such that $x + y + z = xy + yz + zx$. Prove that

$$\frac{x}{\sqrt{x^4+x^2+1}} + \frac{y}{\sqrt{y^4+y^2+1}} + \frac{z}{\sqrt{z^4+z^2+1}} \geq \frac{-1}{\sqrt{3}}.$$

Proposed by Navid Safaei
