VMO 2019
www.artofproblemsolving.com/community/c861220
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Day 1 Let $f:\mathbb{R}\to(0;+\infty)$ be a continuous function such that $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 0.$
a) Prove that $f(x)$ has the maximum value on $\mathbb{R}$.
b) Prove that there exist two sequeneces $(x_n),(y_n)$ with $x_n<y_n, \forall n=1,2,3,...$ such that they have the same limit when n tends to infinity and $f(x_n)=f(y_n)$ for all $n.$

Day 1 Let (x_n) be an integer sequence such that $0\le x_0<x_1\le 100$ and

$$x_{n+2}=7x_{n+1}-x_n+280,\ \forall n\ge 0.$$

a) Prove that if $x_0=2, x_1=3$ then for each positive integer n, the sum of divisors of the following number is divisible by 24

$$x_nx_{n+1}+x_{n+1}x_{n+2}+x_{n+2}x_{n+3}+2018.$$

b) Find all pairs of numbers (x_0,x_1) such that $x_nx_{n+1}+2019$ is a perfect square for infinitely many nonnegative integer numbers $n.$

Day 1 For each real coefficient polynomial $f(x)=a_0+a_1x+\cdots+a_nx^n$, let

$$\Gamma(f(x))=a_0^2+a_1^2+\cdots+a_m^2.$$

Let be given polynomial $P(x)=(x+1)(x+2)\ldots(x+2020)$. Prove that there exists at least 2019 pairwise distinct polynomials $Q_k(x)$ with $1\le k\le 2^{2019}$ and each of it satisfies two following conditions:
i) $\deg Q_k(x)=2020.$
ii) $\Gamma\left(Q_k(x)^n\right)=\Gamma\left(P(x)^n\right)$ for all positive initeger $n.$

Day 1 Let ABC be triangle with H is the orthocenter and I is incenter. Denote A_1,A_2,B_1,B_2,C_1,C_2 be the points on the rays AB,AC,BC,CA,CB, respectively such that

$$AA_1=AA_2=BC, BB_1=BB_2=CA, CC_1=CC_2=AB.$$

Suppose that B_1B_2 cuts C_1C_2 at A', C_1C_2 cuts A_1A_2 at B' and A_1A_2 cuts B_1B_2 at C'.
a) Prove that area of triangle $A'B'C'$ is smaller than or equal to the area of triangle ABC.
b) Let J be circumcenter of triangle $A'B'C'$. AJ cuts BC at R, BJ cuts CA at S and CJ cuts AB at T. Suppose that $(AST),(BTR),(CRS)$ intersect at K. Prove that if triangle ABC is not isosceles then $HIJK$ is a parallelogram.

Day 2 Consider polynomial $f(x) = x^2 - \alpha x + 1$ with $\alpha \in \mathbb{R}$.
a) For $\alpha = \frac{\sqrt{15}}{2}$, let write $f(x)$ as the quotient of two polynomials with nonnegative coefficients.
b) Find all value of α such that $f(x)$ can be written as the quotient of two polynomials with nonnegative coefficients.

Day 2 Let ABC be an acute, nonisosceles triangle with inscribe in a circle (O) and has orthocenter H. Denote M, N, P as the midpoints of sides BC, CA, AB and D, E, F as the feet of the altitudes from vertices A, B, C of triangle ABC. Let K as the reflection of H through BC. Two lines DE, MP meet at X; two lines DF, MN meet at Y.
a) The line XY cut the minor arc BC of (O) at Z. Prove that K, Z, E, F are concyclic.
b) Two lines KE, KF cuts (O) second time at S, T. Prove that BS, CT, XY are concurrent.

Day 2 There are some papers of the size 5×5 with two sides which are divided into unit squares for both sides. One uses n colors to paint each cell on the paper, one cell by one color, such that two cells on the same positions for two sides are painted by the same color. Two painted papers are consider as the same if the color of two corresponding cells are the same. Prove that there are no more than

$$\frac{1}{8}\left(n^{25} + 4n^{15} + n^{13} + 2n^7\right)$$

pairwise distinct papers that painted by this way.
