USAMO 2019

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–	**Day 1** April 17

1 Let $\mathbb{N}$ be the set of positive integers. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies the equation

$$\underbrace{f(f(\ldots f}_{f(n)\text{ times}}(n)\ldots)) = \frac{n^2}{f(f(n))}$$

for all positive integers n. Given this information, determine all possible values of $f(1000)$.

Proposed by Evan Chen

2 Let $ABCD$ be a cyclic quadrilateral satisfying $AD^2 + BC^2 = AB^2$. The diagonals of $ABCD$ intersect at E. Let P be a point on side $\overline{AB}$ satisfying $\angle APD = \angle BPC$. Show that line PE bisects $\overline{CD}$.

Proposed by Ankan Bhattacharya

3 Let K be the set of all positive integers that do not contain the digit 7 in their base-10 representation. Find all polynomials f with nonnegative integer coefficients such that $f(n) \in K$ whenever $n \in K$.

Proposed by Titu Andreescu, Cosmin Pohoata, and Vlad Matei

– **Day 2** April 18

4 Let n be a nonnegative integer. Determine the number of ways that one can choose $(n+1)^2$ sets $S_{i,j} \subseteq \{1, 2, \ldots, 2n\}$, for integers i, j with $0 \le i, j \le n$, such that:

- for all $0 \le i, j \le n$, the set $S_{i,j}$ has $i + j$ elements; and
- $S_{i,j} \subseteq S_{k,l}$ whenever $0 \le i \le k \le n$ and $0 \le j \le l \le n$.

Proposed by Ricky Liu

5 Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Proposed by Yannick Yao

6 Find all polynomials P with real coefficients such that

$$\frac{P(x)}{yz}+\frac{P(y)}{zx}+\frac{P(z)}{xy}=P(x-y)+P(y-z)+P(z-x)$$

holds for all nonzero real numbers x, y, z satisfying $2xyz = x + y + z$.

Proposed by Titu Andreescu and Gabriel Dospinescu