The problems from the 26th Macedonian Mathematical Olympiad

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1 In an acute-angled triangle ABC, point M is the midpoint of side BC and the centers of the M-excircles of triangles AMB and AMC are D and E, respectively. The circumcircle of triangle ABD intersects line BC at points B and F. The circumcircle of triangle ACE intersects line BC at points C and G. Prove that $BF = CG$.

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2 Let n be a positive integer. If $r \equiv \ n \ (mod \ 2)$ and $r \in \{0, 1\}$, find the number of integer solutions to the system of equations

$$\begin{cases} x + y + z = r \\ \mid x \mid + \mid y \mid + \mid z \mid = n \end{cases}$$

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3 Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

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4 Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$n! \ + \ f(m)! \mid f(n)! \ + \ f(m!)$,

for all $m, n \in \mathbb{N}$.

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5 Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.
Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)
