Czech-Polish-Slovak Match 2014

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by parmenides51

– Day 1

1 Prove that if the positive real numbers a, b, c satisfy the equation

$$a^4 + b^4 + c^4 + 4a^2b^2c^2 = 2(a^2b^2 + a^2c^2 + b^2c^2),$$

then there is a triangle ABC with internal angles α, β, γ such that

$$\sin\alpha = a, \qquad \sin\beta = b, \qquad \sin\gamma = c.$$

2 For the positive integers a, b, x_1 we construct the sequence of numbers $(x_n)_{n=1}^{\infty}$ such that $x_n = ax_{n-1} + b$ for each $n \geq 2$. Specify the conditions for the given numbers a, b and x_1 which are necessary and sufficient for all indexes m, n to apply the implication $m|n \Rightarrow x_m|x_n$.

(Jaromír Šimša)

3 Given is a convex $ABCD$, which is $|\angle ABC| = |\angle ADC| = 135^\circ$. On the AB, AD are also selected points M, N such that $|\angle MCD| = |\angle NCB| = 90^\circ$. The circumcircles of the triangles AMN and ABD intersect for the second time at point $K \neq A$. Prove that lines AK and KC are perpendicular.

(Irán)

– Day 2

4 Let ABC be a triangle, and let P be the midpoint of AC. A circle intersects AP, CP, BC, AB sequentially at their inner points K, L, M, N. Let S be the midpoint of KL. Let also $2 \cdot |AN| \cdot |AB| \cdot |CL| = 2 \cdot |CM| \cdot |BC| \cdot |AK| = |AC| \cdot |AK| \cdot |CL|$. Prove that if $P \neq S$, then the intersection of KN and ML lies on the perpendicular bisector of the PS.

(Jan Mazák)

5 Let all positive integers n satisfy the following condition:
for each non-negative integers k, m with $k + m \leq n$,
the numbers $\binom{n-k}{m}$ and $\binom{n-m}{k}$ leave the same remainder when divided by 2.

(Poland)

PS. The translation was done using Google translate and in case it is not right, there is the original text in Slovak

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6 Let $n \geq 6$ be an integer and F be the system of the 3-element subsets of the set $\{1, 2, ..., n\}$ satisfying the following condition:
for every $1 \leq i < j \leq n$ there is at least $\lfloor \frac{1}{3}n \rfloor - 1$ subsets $A \in F$ such that $i, j \in A$.
Prove that for some integer $m \geq 1$ exist the mutually disjoint subsets $A_1, A_2, ..., A_m \in F$ also, that $|A_1 \cup A_2 \cup ... \cup A_m| \geq n - 5$

(Poland)

PS. just in case my translation does not make sense,
I leave the original in Slovak, in case someone understands something else

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