Sharygin Geometry Olympiad 2005

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by parmenides51

– First (Correspondence) Round

1 The chords AC and BD of the circle intersect at point P. The perpendiculars to AC and BD at points C and D, respectively, intersect at point Q. Prove that the lines AB and PQ are perpendicular.

2 Cut a cross made up of five identical squares into three polygons, equal in area and perimeter.

3 Given a circle and a point K inside it. An arbitrary circle equal to the given one and passing through the point K has a common chord with the given circle. Find the geometric locus of the midpoints of these chords.

4 At what smallest n is there a convex n-gon for which the sines of all angles are equal and the lengths of all sides are different?

5 There are two parallel lines p_1 and p_2. Points A and B lie on p_1, and C on p_2. We will move the segment BC parallel to itself and consider all the triangles $AB'C'$ thus obtained. Find the locus of the points in these triangles:
a) points of intersection of heights,
b) the intersection points of the medians,
c) the centers of the circumscribed circles.

6 Side AB of triangle ABC was divided into n equal parts (dividing points $B_0 = A, B_1, B_2, ..., B_n = B$), and side AC of this triangle was divided into $(n + 1)$ equal parts (dividing points $C_0 = A, C_1, C_2, ..., C_{n+1} = C$). Colored are the triangles $C_iB_iC_{i+1}$ (where $i = 1, 2, ..., n$). What part of the area of the triangle is painted over?

7 Two circles with radii 1 and 2 have a common center at the point O. The vertex A of the regular triangle ABC lies on the larger circle, and the middpoint of the base CD lies on the smaller one. What can the angle BOC be equal to?

8 Around the convex quadrilateral $ABCD$, three rectangles are circumscribed .
It is known that two of these rectangles are squares. Is it true that the third one is necessarily a square?
(A rectangle is circumscribed around the quadrilateral $ABCD$ if there is one vertex $ABCD$ on each side of the rectangle).

9 Let O be the center of a regular triangle ABC. From an arbitrary point P of the plane, the perpendiculars were drawn on the sides of the triangle. Let M denote the intersection point of the medians of the triangle , having vertices the feet of the perpendiculars. Prove that M is the midpoint of the segment PO.

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10 Cut the non-equilateral triangle into four similar triangles, among which not all are the same.

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11 The square was cut into n^2 rectangles with sides $a_i \times b_j, i, j = 1, ..., n.$
For what is the smallest n in the set $\{a_1, b_1, ..., a_n, b_n\}$ all the numbers can be different?

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12 Construct a quadrangle along the given sides a, b, c, and d and the distance I between the midpoints of its diagonals.

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13 A triangle ABC and two lines ℓ_1, ℓ_2 are given. Through an arbitrary point D on the side AB, a line parallel to ℓ_1 intersects the AC at point E and a line parallel to ℓ_2 intersects the BC at point F. Construct a point D for which the segment EF has the smallest length.

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14 Let P be an arbitrary point inside the triangle ABC. Let A_1, B_1 and C_1 denote the intersection points of the straight lines AP, BP and CP, respectively, with the sides BC, CA and AB. We order the areas of the triangles $AB_1C_1, A_1BC_1, A_1B_1C$. Denote the smaller by S_1, the middle by S_2, and the larger by S_3. Prove that $\sqrt{S_1S_2} \le S \le \sqrt{S_2S_3}$,where S is the area of the triangle $A_1B_1S_1$.

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15 Given a circle centered at the origin.
Prove that there is a circle of smaller radius that has no less points with integer coordinates.

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16 We took a non-equilateral acute-angled triangle and marked 4 wonderful points in it: the centers of the inscribed and circumscribed circles, the center of gravity (the point of intersection of the medians) and the intersection point of altitudes. Then the triangle itself was erased. It turned out that it was impossible to establish which of the centers corresponds to each of the marked points. Find the angles of the triangle

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17 A circle is inscribed in the triangle ABC and it's center I and the points of tangency P, Q, R with the sides BC, CA and AB are marked, respectively. With a single ruler, build a point K at which the circle passing through the vertices B and C touches (internally) the inscribed circle.

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18 On the plane are three straight lines ℓ_1, ℓ_2, ℓ_3, forming a triangle, and the point O is marked, the center of the circumscribed circle of this triangle. For an arbitrary point X of the plane, we denote by X_i the point symmetric to the point X with respect to the line $\ell_i, i = 1, 2, 3.$
a) Prove that for an arbitrary point M the straight lines connecting the midpoints of the segments O_1O_2 and M_1M_2, O_2O_3 and M_2M_3, O_3O_1 and M_3M_1 intersect at one point,
b) where can this intersection point lie?

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19 As you know, the moon revolves around the earth. We assume that the Earth and the Moon are points, and the Moon rotates around the Earth in a circular orbit with a period of one revolution per month.
The flying saucer is in the plane of the lunar orbit. It can be jumped through the Moon and the Earth - from the old place (point A), it instantly appears in the new (at point A') so that either the Moon or the Earth is in the middle of segment AA'. Between the jumps, the flying saucer hangs motionless in outer space.
1) Determine the minimum number of jumps a flying saucer will need to jump from any point inside the lunar orbit to any other point inside the lunar orbit.
2) Prove that a flying saucer, using an unlimited number of jumps, can jump from any point inside the lunar orbit to any other point inside the lunar orbit for any period of time, for example, in a second.

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20 Let I be the center of the sphere inscribed in the tetrahedron $ABCD, A', B', C', D'$ be the centers of the spheres circumscribed around the tetrahedra $IBCD, ICDA, IDAB, IABC$, respectively. Prove that the sphere circumscribed around $ABCD$ lies entirely inside the circumscribed around $A'B'C'D'$.

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21 The planet Tetraincognito covered by ocean has the shape of a regular tetrahedron with an edge of 900 km. What area of the ocean will the tsunami' cover 2 hours after the earthquake with the epicenter in
a) the center of the face,
b) the middle of the edge,
if the tsunami propagation speed is 300 km / h?

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22 Perpendiculars at their centers of gravity (points of intersection of medians) are restored to the faces of the tetrahedron. Prove that the projections of the three perpendiculars to the fourth face intersect at one point.

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23 Envelop the cube in one layer with five convex pentagons of equal areas.

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24 A triangle is given, all the angles of which are smaller than ϕ, where $\phi < 2\pi/3$. Prove that in space there is a point from which all sides of the triangle are visible at an angle ϕ.

– Final Round

– grade IX

9.1 The quadrangle $ABCD$ is inscribed in a circle whose center O lies inside it.
Prove that if $\angle BAO = \angle DAC$, then the diagonals of the quadrilateral are perpendicular.

9.2 Find all isosceles triangles that cannot be cut into three isosceles triangles with the same sides.

9.3 Given a circle and points A, B on it. Draw the set of midpoints of the segments, one of the ends of which lies on one of the arcs AB, and the other on the second.

9.4 Let P be the intersection point of the diagonals of the quadrangle $ABCD$, M the intersection point of the lines connecting the midpoints of its opposite sides, O the intersection point of the perpendicular bisectors of the diagonals, H the intersection point of the lines connecting the orthocenters of the triangles APD and BCP, APB and CPD. Prove that M is the midpoint of OH.

9.5 It is given that for no side of the triangle from the height drawn to it, the bisector and the median it is impossible to make a triangle. Prove that one of the angles of the triangle is greater than 135^o

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– grade X

10.1 A convex quadrangle without parallel sides is given. For each triple of its vertices, a point is constructed that supplements this triple to a parallelogram, one of the diagonals of which coincides with the diagonal of the quadrangle. Prove that of the four points constructed, exactly one lies inside the original quadrangle.

10.2 A triangle can be cut into three similar triangles.
Prove that it can be cut into any number of triangles similar to each other.

10.3 Two parallel chords AB and CD are drawn in a circle with center O.
Circles with diameters AB and CD intersect at point P.
Prove that the midpoint of the segment OP is equidistant from lines AB and CD.

10.4 Two segments A_1B_1 and A_2B_2 are given on the plane, with $\frac{A_2B_2}{A_1B_1} = k < 1$. On segment A_1A_2, point A_3 is taken, and on the extension of this segment beyond point A_2, point A_4 is taken, so $\frac{A_3A_2}{A_3A_1} = \frac{A_4A_2}{A_4A_1} = k$. Similarly, point B_3 is taken on segment B_1B_2 , and on the extension of this the segment beyond point B_2 is point B_4, so $\frac{B_3B_2}{B_3B_1} = \frac{B_4B_2}{B_4B_1} = k$. Find the angle between lines A_3B_3 and A_4B_4.

(Netherlands)

10.5 Two circles of radius 1 intersect at points X, Y, the distance between which is also equal to 1. From point C of one circle, tangents CA, CB are drawn to the other. Line CB will cross the first circle a second time at point A'. Find the distance AA'.

10.6 Let H be the orthocenter of triangle ABC, X be an arbitrary point. A circle with a diameter of XH intersects lines AH, BH, CH at points A_1, B_1, C_1 for the second time, and lines $AXBX, CX$ at points A_2, B_2, C_2. Prove that lines A$_1A_2, B_1B_2, C_1C_2$ intersect at one point.

– grade XI

11.1 A_1, B_1, C_1 are the midpoints of the sides BC, CA, BA respectively of an equilateral triangle ABC. Three parallel lines, passing through A_1, B_1, C_1 intersect, respectively, lines B_1C_1, C_1A_1, A_1B_1 at points A_2, B_2, C_2. Prove that the lines AA_2, BB_2, CC_2 intersect at one point lying on the circle circumscribed around the triangle ABC.

11.2 Convex quadrilateral $ABCD$ is given. Lines BC and AD intersect at point O, with B lying on the segment OC, and A on the segment OD. I is the center of the circle inscribed in the OAB triangle, J is the center of the circle exscribed in the triangle OCD touching the side of CD and the extensions of the other two sides. The perpendicular from the midpoint of the segment IJ on the lines BC and AD intersect the corresponding sides of the quadrilateral (not the extension) at points X and Y. Prove that the segment XY divides the perimeter of the quadrilateral$ABCD$ in half, and from all segments with this property and ends on BC and AD, segment XY has the smallest length.

11.3 Inside the inscribed quadrilateral $ABCD$ there is a point K, the distances from which to the sides $ABCD$ are proportional to these sides. Prove that K is the intersection point of the diagonals of $ABCD$.

11.4 In the triangle $ABC, \angle A = \alpha, BC = a$. The inscribed circle touches the lines AB and AC at points M and P. Find the length of the chord cut by the line MP in a circle with diameter BC.

11.5 The angle and the point K inside it are given on the plane. Prove that there is a point M with the following property:
if an arbitrary line passing through intersects the sides of the angle at points A and B, then MK is the bisector of the angle AMB.

11.6 The sphere inscribed in the tetrahedron $ABCD$ touches its faces at points A', B', C', D'. The segments AA' and BB' intersect, and the point of their intersection lies on the inscribed sphere. Prove that the segments CC' and DD' also intersect on the inscribed sphere.