Sharygin Geometry Olympiad 2006

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by parmenides51

– First (Correspodence) Round

1 Two straight lines intersecting at an angle of 46^o are the axes of symmetry of the figure F on the plane. What is the smallest number of axes of symmetry this figure can have?

2 Points A, B move with equal speeds along two equal circles.
Prove that the perpendicular bisector of AB passes through a fixed point.

3 The map shows sections of three straight roads connecting the three villages, but the villages themselves are located outside the map. In addition, the fire station located at an equal distance from the three villages is not indicated on the map, although its location is within the map. Is it possible to find this place with the help of a compass and a ruler, if the construction is carried out only within the map?

4 a) Given two squares $ABCD$ and $DEFG$, with point E lying on the segment CD, and pointsF, G outside the square $ABCD$. Find the angle between lines AE and BF.
b) Two regular pentagons $OKLMN$ and $OPRST$ are given, and the point P lies on the segment ON, and the points R, S, T are outside the pentagon $OKLMN$. Find the angle between straight lines KP and MS.

5 a) Fold a 10×10 square from a 1×118 rectangular strip.
b) Fold a 10×10 square from a $1 \times (100 + 9\sqrt{3})$ rectangular strip (approximately 1×115.58).
The strip can be bent, but not torn.

6 a) Given a segment AB with a point C inside it, which is the chord of a circle of radius R.
Inscribe in the formed segment a circle tangent to point C and to the circle of radius R.
b) Given a segment AB with a point C inside it, which is the point of tangency of a circle of radius r.
Draw through A and B a circle tangent to a circle of radius r.

7 The point E is taken inside the square $ABCD$, the point F is taken outside, so that the triangles ABE and BCF are congruent . Find the angles of the triangle ABE, if it is known thatEF is equal to the side of the square, and the angle BFD is right.

8 The segment AB divides the square into two parts, in each of which a circle can be inscribed. The radii of these circles are equal to r_1 and r_2 respectively, where $r_1 > r_2$.
Find the length of AB.

9 $L(a)$ is the line connecting the points of the unit circle corresponding to the angles a and $\pi - 2a$. Prove that if $a + b + c = 2\pi$, then the lines $L(a), L(b)$ and $L(c)$ intersect at one point.

10 At what n can a regular n-gon be cut by disjoint diagonals into $n - 2$ isosceles (including equilateral) triangles?

11 In the triangle ABC, O is the center of the circumscribed circle, A', B', C' are the symmetrics of A, B, C with respect to opposite sides, A_1, B_1, C_1 are the intersection points of the lines OA' and BC, OB' and AC, OC' and AB. Prove that the lines AA_1, BB_1, CC_1 intersect at one point.

12 In the triangle ABC, the bisector of angle A is equal to the half-sum of the height and median drawn from vertex A. Prove that if $\angle A$ is obtuse, then $AB = AC$.

13 Two straight lines a and b are given and also points A and B. Point X slides along the line a, and point Y slides along the line b, so that $AX \parallel BY$. Find the locus of the intersection point of AY with XB.

14 Given a circle and a fixed point P not lying on it. Find the geometrical locus of the orthocenters of the triangles ABP, where AB is the diameter of the circle.

15 A circle is circumscribed around triangle ABC and a circle is inscribed in it, which touches the sides of the triangle BC, CA, AB at points A_1, B_1, C_1, respectively. The line B_1C_1 intersects the line BC at the point P, and M is the midpoint of the segment PA_1. Prove that the segments of the tangents drawn from the point M to the inscribed and circumscribed circle are equal.

16 Regular triangles are built on the sides of the triangle ABC. It turned out that their vertices form a regular triangle. Is the original triangle regular also?

17 In two circles intersecting at points A and B, parallel chords A_1B_1 and A_2B_2 are drawn. The lines AA_1 and BB_2 intersect at the point X, AA_2 and BB_1 intersect at the point Y. Prove that $XY//A_1B_1$.

18 Two perpendicular lines are drawn through the orthocenter H of triangle ABC, one of which intersects BC at point X, and the other intersects AC at point Y. Lines AZ, BZ are parallel, respectively with HX and HY. Prove that the points X, Y, Z lie on the same line.

19 Through the midpoints of the sides of the triangle T, straight lines are drawn perpendicular to the bisectors of the opposite angles of the triangle. These lines formed a triangle T_1. Prove that the center of the circle circumscribed about T_1 is in the midpoint of the segment formed by the center of the inscribed circle and the intersection point of the heights of triangle T.

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20 Four points are given A, B, C, D. Points A_1, B_1, C_1, D_1 are orthocenters of the triangles BCD, CDA, DAB, A and A_2, B_2, C_2, D_2 are orthocenters of the triangles $B_1C_1D_1, C_1D_1A_1, D_1A_1B_1, A_1B_1C_1$ etc. Prove that the circles passing through the midpoints of the sides of all the triangles intersect at one point.

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21 On the sides AB, BC, CA of triangle ABC, points C', A', B' are taken.
Prove that for the areas of the corresponding triangles, the inequality holds:

$$S_{ABC}S^2_{A'B'C'} \geq 4S_{AB'C'}S_{BC'A'}S_{CA'B'}$$

and equality is achieved if and only if the lines AA', BB', CC' intersect at one point.

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22 Given points A, B on a circle and a point P not lying on the circle. X is an arbitrary point of the circle, Y is the intersection point of lines AX and BP. Find the locus of the centers of the circles circumscribed around the triangles PXY.

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23 $ABCD$ is a convex quadrangle, G is its center of gravity as a homogeneous plate (i.e., the intersection point of two lines, each of which connects the centroids of triangles having a common diagonal).
a) Suppose that around $ABCD$ we can circumscribe a circle centered on O. We define H similarly to G, taking orthocenters instead of centroids. Then the points of H, G, O lie on the same line and $HG : GO = 2 : 1$.
b) Suppose that in $ABCD$ we can inscribe a circle centered on I. The Nagel point N of the circumscribed quadrangle is the intersection point of two lines, each of which passes through points on opposite sides of the quadrangle that are symmetric to the tangent points of the inscribed circle relative to the midpoints of the sides. (These lines divide the perimeter of the quadrangle in half). Then N, G, I lie on one straight line, with $NG : GI = 2 : 1$.

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24 a) Two perpendicular rays are drawn through a fixed point P inside a given circle, intersecting the circle at points A and B. Find the geometric locus of the projections of P on the lines AB.
b) Three pairwise perpendicular rays passing through the fixed point P inside a given sphere intersect the sphere at points A, B, C. Find the geometrical locus of the projections P on the ABC plane

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25 In the tetrahedron $ABCD$, the dihedral angles at the BC, CD, and DA edges are equal to α, and for the remaining edges equal to β. Find the ratio AB/CD.

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26 Four cones are given with a common vertex and the same generatrix, but with, generally speaking, different radii of the bases. Each of them is tangent to two others. Prove that the four tangent points of the circles of the bases of the cones lie on the same circle.

– Final Round

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– grade VIII

8.1 Inscribe the equilateral triangle of the largest perimeter in a given semicircle.

8.2 What n is the smallest such that "there is a n-gon that can be cut into a triangle, a quadrilateral, ..., a 2006-gon"?

8.3 A parallelogram $ABCD$ is given. Two circles with centers at the vertices A and C pass through B. The straight line ℓ that passes through B and crosses the circles at second time at points X, Y respectively. Prove that $DX = DY$.

8.4 Two equal circles intersect at points A and B. P is the point of one of the circles that is different from A and B, X and Y are the second intersection points of the lines of PA, PB with the other circle. Prove that the line passing through P and perpendicular to AB divides one of the arcs XY in half.

8.5 Is there a convex polygon with each side equal to some diagonal, and each diagonal equal to some side?

8.6 A triangle ABC and a point P inside it are given. A', B', C' are the projections of P onto the straight lines ot the sides BC, CA, AB. Prove that the center of the circle circumscribed around the triangle $A'B'C'$ lies inside the triangle ABC.

– grade IX

9.1 Given a circle of radius K. Two other circles, the sum of the radii of which are also equal to K, tangent to the circle from the inside. Prove that the line connecting the points of tangency passes through one of the common points of these circles.

9.2 Given a circle, point A on it and point M inside it. We consider the chords BC passing through M. Prove that the circles passing through the midpoints of the sides of all the triangles ABC are tangent to a fixed circle.

9.3 Triangles ABC and $A_1B_1C_1$ are similar and differently oriented. On the segment AA_1, a point A' is taken such that $AA'/A_1A' = BC/B_1C_1$. We similarly construct B' and C'. Prove that A', B', C' lie on one straight line.

9.4 In a non-convex hexagon, each angle is either 90 or 270 degrees. Is it true that for some lengths of the sides it can be cut into two hexagons similar to it and unequal to each other?

9.5 A straight line passing through the center of the circumscribed circle and the intersection point of the heights of the non-equilateral triangle ABC divides its perimeter and area in the same ratio.Find this ratio.

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9.6 A convex quadrilateral ABC is given. A', B', C', D' are the orthocenters of triangles BCD, CDA, DAB, ABC respectively. Prove that in the quadrilaterals $ABCP$ and $A'B'C'D'$, the corresponding diagonals share the intersection points in the same ratio.

– grade X

10.1 Five lines go through one point. Prove that there exists a closed five-segment polygonal line, the vertices and the middle of the segments of which lie on these lines, and each line has exactly one vertex.

10.2 The projections of the point X onto the sides of the $ABCD$ quadrangle lie on the same circle. Y is a point symmetric to X with respect to the center of this circle. Prove that the projections of the point B onto the lines AX, XC, CY, YA also lie on the same circle.

10.3 Given a circle and a point P inside it, different from the center. We consider pairs of circles tangent to the given internally and to each other at point P. Find the locus of the points of intersection of the common external tangents to these circles.

10.4 Lines containing the medians of the triangle ABC intersect its circumscribed circle for a second time at the points A_1, B_1, C_1. The straight lines passing through A, B, C parallel to opposite sides intersect it at points A_2, B_2, C_2. Prove that lines A_1A_2, B_1B_2, C_1C_2 intersect at one point.

10.5 Can a tetrahedron scan turn out to be a triangle with sides $3, 4$ and 5 (a tetrahedron can be cut only along the edges)?

10.6 A quadrangle was drawn on the board, that you can inscribe and circumscribe a circle. Marked are the centers of these circles and the intersection point of the lines connecting the midpoints of the opposite sides, after which the quadrangle itself was erased. Restore it with a compass and ruler.