I Caucasus Mathematical Olympiad

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by parmenides51

–	grade 7	
1	Does there exist a four-digit positive integer with different non-zero digits, which has the following property: if we add the same number written in the reverse order, then we get a number divisible by 101?	
2	There are 9 cards with the numbers $1, 2, 3, 4, 5, 6, 7, 8$ and 9. What is the largest number of these cards can be decomposed in a certain order in a row, so that in any two adjacent cards, one of the numbers is divided by the other?	
3	Petya bought one cake, two cupcakes and three bagels, Apya bought three cakes and a bagel, and Kolya bought six cupcakes. They all paid the same amount of money for purchases. Lena bought two cakes and two bagels. And how many cupcakes could be bought for the same amount spent to her?	
4	There are 26 students in the class. They agreed that each of them would either be a liar (liars always lie) or a knight (knights always tell the truth). When they came to the class and sat down for desks, each of them said: "I am sitting next to a liar." Then some students moved for other desks. After that, everyone says: " I am sitting next to a knight ." Is this possible? Every time exactly two students sat at any desk.	
5	What is the smallest number of 3-cell corners needed to be painted in a 6×6 square so that it was impossible to paint more than one corner of it? (The painted corners should not overlap.)	
–	grade 8	
1	Find some four different natural numbers with the following property: if you add to the product of any two of them the product of the two remaining numbers. you get a prime number.	
2	In the convex quadrilateral $ABCD$, point K is the midpoint of AB, point L is the midpoint of BC, point M is the midpoint of CD, and point N is the midpoint of DA. Let S be a point lying inside the quadrilateral $ABCD$ such that $KS = LS$ and $NS = MS$ .Prove that $\angle KSN = \angle MSL$.	

3 The workers laid a floor of size $n \times n$ with tiles of two types: 2×2 and 3×1.
It turned out that they were able to completely lay the floor in such a way that the same number of tiles of each type was used. Under what conditions could this happen?
(You can't cut tiles and also put them on top of each other.)

4 The sum of the numbers a, b and c is zero, and their product is negative.
Prove that the number $\frac{a^2+b^2}{c} + \frac{b^2+c^2}{a} + \frac{c^2+a^2}{b}$ is positive.

5 On the table are 300 coins. Petya, Vasya and Tolya play the next game. They go in turn in the following order: Petya, Vasya, Tolya, Petya. Vasya, Tolya, etc. In one move, Petya can take $1, 2, 3$, or 4 coins from the table, Vasya, 1 or 2 coins, and Tolya, too, 1 or 2 coins. Can Vasya and Tolya agree so that, as if Petya were playing, one of them two will take the last coin off the table?

– grade 9

1 At the round table, 10 people are sitting, some of them are knights, and the rest are liars (knights always say pride, and liars always lie) . It is clear thath I have at least one knight and at least one liar.
What is the largest number of those sitting at the table can say: "Both of my neighbors are knights " ?
(A statement that is at least partially false is considered false.)

2 Let a and b be arbitrary distinct numbers.
Prove that the equation $(x+a)(x+b) = 2x+a+b$ has two different roots.

3 Let AL be the angle bisector of the acute-angled triangle ABC. and ω be the circle circumscribed about it. Denote by P the intersection point of the extension of the altitude BH of the triangle ABC with the circle ω . Prove that if $\angle BLA = \angle BAC$, then $BP = CP$.

4 Is there a nine-digit number without zero digits, the remainder of dividing which on each of its digits is different?

5 Let's call a natural number a palindrome, the decimal notation of which is equally readable from left to right and right to left (decimal notation cannot start from zero; for example, the number 1221 is a palindrome, but the numbers $1231, 1212$ and 1010 are not). Which palindromes among the numbers from $10,000$ to $999,999$ have an odd sum of digits, which have an one even, and how many times are the ones with odd sum more than the ones with the even sum?

– grade 10

1 Find the roots of the equation $(x-a)(x-b) = (x-c)(x-d)$, if you know that $a+d = b+c = 2015$ and $a \neq c$ (numbers a, b, c, d are not given).

2 Vasya chose a certain number x and calculated the following: $a_1 = 1 + x^2 + x^3, a_2 = 1 + x^3 + x^4, a_3 = 1 + x^4 + x^5, ..., a_n = 1 + x^{n+1} + x^{n+2}, ...$
It turned out that $a_2^2 = a_1 a_3$.
Prove that for all $n \geq 3$, the equality $a_n^2 = a_{n-1} a_{n+1}$ holds.

3 What is the smallest number of 3-cell corners that you need to paint in a 5×5 square so that you cannot paint more than one corner of one it? (Shaded corners should not overlap.)

4 We call a number greater than 25, *semi-prime* if it is the sum of some two different prime numbers. What is the greatest number of consecutive natural numbers that can be *semi-prime*?

5 Let AA_1 and CC_1 be the altitudes of the acute-angled triangle ABC. Let K, L and M be the midpoints of the sides AB, BC and CA respectively. Prove that if $\angle C_1MA_1 = \angle ABC$, then $C_1K = A_1L$.

– grade 11

1 Is there an eight-digit number without zero digits, which when divided by the first digit gives the remainder 1, when divided by the second digit will give the remainder 2, ..., when divided by the eighth digit will give the remainder 8?

2 The equation $(x + a)(x + b) = 9$ has a root $a + b$. Prove that $ab \leq 1$.

3 The workers laid a floor of size $n \times n$ ($10 < n < 20$) with two types of tiles: 2×2 and 5×1. It turned out that they were able to completely lay the floor so that the same number of tiles of each type was used. For which n could this happen? (You can't cut tiles and also put them on top of each other.)

4 The midpoint of the edge SA of the triangular pyramid of $SABC$ has equal distances from all the vertices of the pyramid. Let SH be the height of the pyramid. Prove that $BA^2 + BH^2 = CA^2 + CH^2$.

5 Are there natural $a, b > 1000$, such that for any c that is a perfect square, the three numbers a, b and c are not the lengths of the sides of a triangle?