IMOR 2018

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by math_roots

1 Let p be a prime number, and X be the set of cubes modulo p, including 0. Denote by $C_2(k)$ the number of ordered pairs $(x, y) \in X \times X$ such that $x + y \equiv k \pmod p$. Likewise, denote by $C_3(k)$ the number of ordered pairs $(x, y, z) \in X \times X \times X$ such that $x + y + z \equiv k \pmod p$. Prove that there are integers a, b such that for all k not in X, we have

$$C_3(k) = a \cdot C_2(k) + b.$$

Proposed by Murilo Corato, Brazil.

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2 Let G be the centroid of a triangle $\triangle ABC$ and let AG, BG, CG meet its circumcircle at P, Q, R respectively.
Let AD, BE, CF be the altitudes of the triangle. Prove that the radical center of circles $(DQR), (EPR), (FPQ)$ lies on Euler Line of $\triangle ABC$.

Proposed by Ivan Chai, Malaysia.

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3 When the IMO is over and students want to relax, they all do the same thing: download movies from the internet. There is a positive number of rooms with internet routers at the hotel, and each student wants to download a positive number of bits. The load of a room is defined as the total number of bits to be downloaded from that room. Nobody likes slow internet, and in particular each student has a displeasure equal to the product of her number of bits and the load of her room. The misery of the group is defined as the sum of the students displeasures.

Right after the contest, students gather in the hotel lobby to decide who goes to which room. After much discussion they reach a balanced configuration: one for which no student can decrease her displeasure by unilaterally moving to another room. The misery of the group is computed to be M_1, and right when they seemed satisfied, Gugu arrived with a serendipitous smile and proposed another configuration that achieved misery M_2. What is the maximum value of M_1/M_2 taken over all inputs to this problem?

Proposed by Victor Reis (proglote), Brazil.

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4 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x)^2 - f(y)^2 = f(x + y) \cdot f(x - y),$$

for all $x, y \in \mathbb{Q}$.

Proposed by Portugal.