Tuymaada Olympiad 1996

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by parmenides51

– day 1

1 Prove the inequality $x_1y_1 + x_2y_2 + x_2y_1 + 2x_2y_2 \leq 1996$
if $x_1^2 + 2x_1x_2 + 2x_2^2 \leq 998$ and $y_1^2 + 2y_1y_2 + 2y_2^2 \leq 3992$.

2 Given a finite set of real numbers A, not containing 0 and 1 and possessing the property: if the number a belongs to A, then numbers $\frac{1}{a}$ and $1 - a$ also belong to A. How many numbers are in the set A?

3 Nine points of the plane, located at the vertices of a regular nonagon, are pairwise connected by segments, each of which is colored either red or blue. It is known that in any triangle with vertices at the vertices of the nonagon at least one side is red. Prove that there are four points, any two of which are connected by red lines.

4 Given a segment of length $7\sqrt{3}$.
Is it possible to use only compass to construct a segment of length $\sqrt{7}$?

– day 2

5 Solve the equation $\sqrt{1981 - \sqrt{1996 + x}} = x + 15$

6 Given the sequence $f_1(a) = sin(0,5\pi a)$ $f_2(a) = sin(0,5\pi(sin(0,5\pi a)))$ ... $f_n(a) = sin(0,5\pi(sin(...(sin(0,5\pi$
, where a is any real number.
What limit aspire the members of this sequence as $n \to \infty$?

7 In the set of all positive real numbers define the operation $a * b = a^b$.
Find all positive rational numbers for which $a * b = b * a$.

8 Given a tetrahedron $ABCD$, in which $AB = CD = 13, AC = BD = 14$ and $AD = BC = 15$.
Show that the centers of the inscribed sphere and sphere around it coincide, and find the radii of these spheres.