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by ABCDE

1 Let $m > 1$ be a fixed positive integer. For a nonempty string of base-ten digits S, let $c(S)$ be the number of ways to split S into contiguous nonempty strings of digits such that the base-ten number represented by each string is divisible by m. These strings are allowed to have leading zeroes.

In terms of m, what are the possible values that $c(S)$ can take?

For example, if $m = 2$, then $c(1234) = 2$ as the splits 1234 and $12|34$ are valid, while the other six splits are invalid.

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2 Consider a finite set of points $T \in \mathbb{R}^n$ contained in the n-dimensional unit ball centered at the origin, and let X be the convex hull of T. Prove that for all positive integers k and all points $x \in X$, there exist points $t_1, t_2, \ldots, t_k \in T$, not necessarily distinct, such that their centroid

$$\frac{t_1 + t_2 + \cdots + t_k}{k}$$

has Euclidean distance at most $\frac{1}{\sqrt{k}}$ from x.

(The n-dimensional unit ball centered at the origin is the set of points in $\mathbb{R}^n$ with Euclidean distance at most 1 from the origin. The convex hull of a set of points $T \in \mathbb{R}^n$ is the smallest set of points X containing T such that each line segment between two points in X lies completely inside X.)

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3 A polygon in the plane (with no self-intersections) is called *equitable* if every line passing through the origin divides the polygon into two (possibly disconnected) regions of equal area.

Does there exist an equitable polygon which is not centrally symmetric about the origin?

(A polygon is centrally symmetric about the origin if a 180-degree rotation about the origin sends the polygon to itself.)

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4 Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$$

for all $x, y \in \mathbb{R}^+$.

($\mathbb{R}^+$ denotes the set of positive real numbers.)

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5 Let G be an undirected simple graph. Let $f(G)$ be the number of ways to orient all of the edges of G in one of the two possible directions so that the resulting directed graph has no directed cycles. Show that $f(G)$ is a multiple of 3 if and only if G has a cycle of odd length.
