KJMO 2006

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–	day 1
1	$a_1, a_2, ..., a_{2006}$ is a permutation of $1, 2, ..., 2006$. Prove that $\prod_{i=1}^{2006}(a_i^2 - i)$ is a multiple of 3. (0 is counted as a multiple of 3)
2	Find all positive integers that can be written in the following way $\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}$. Also, a, b, c are positive integers that are pairwise relatively prime.
3	In a circle O, there are six points, A, B, C, D, E, F in a counterclockwise order. $BD \perp CF$, and CF, BE, AD are concurrent. Let the perpendicular from B to AC be M, and the perpendicular from D to CE be N. Prove that $AE//MN$.
4	In the coordinate plane, de fine $M = \{(a, b), a, b \in Z\}$. A transformation S, which is de fined on M, sends (a, b) to $(a + b, b)$. Transformation T, also de fined on M, sends (a, b) to $(-b, a)$. Prove that for all $(a, b) \in M$, we can use S, T denitely to map it to $(g, 0)$.
–	day 2
5	Find all positive integers that can be written in the following way $\frac{m^2+20mn+n^2}{m^3+n^3}$ Also, m, n are relatively prime positive integers.
6	For all reals a, b, c, d prove the following inequality: $$\frac{a+b+c+d}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} < 1$$
7	A line through point P outside of circle O meets the said circle at B, C ($PB < PC$). Let PO meet circle O at Q, D (with $PQ < PD$). Let the line passing Q and perpendicular to BC meet circle O at A. If $BD^2 = AD \cdot CP$, prove that PA is a tangent to O.
8	De ne the set F as the following: $F = \{(a_1, a_2, ..., a_{2006}) : \forall i = 1, 2, ..., 2006, a_i \in \{-1, 1\}\}$ Prove that there exists a subset of F, called S which satis es the following: $\lvert S\rvert = 2006$ and for all $(a_1, a_2, ..., a_{2006}) \in F$ there exists $(b_1, b_2, ..., b_{2006}) \in S$, such that $\Sigma_{i=1}^{2006} a_i b_i = 0$.