KJMO 2007

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–	day 1
1	A sequence $a_1, a_2, ..., a_{2007}$ where $a_i \in \{2,3\}$ for $i = 1,2,...,2007$ and an integer sequence $x_1, x_2, ..., x_{2007}$ satis fies the following: $a_i x_i + x_{i+2} \equiv 0 \ (mod 5)$, where the indices are taken modulo 2007. Prove that $x_1, x_2, ..., x_{2007}$ are all multiples of 5.
2	If n is a positive integer and a, b are relatively prime positive integers, calculate $(a+b, a^n+b^n)$.
3	Consider the string of length 6 composed of three characters a, b, c. For each string, if two as are next to each other, or two bs are next to each other, then replace aa by b, and replace bb by a. Also, if a and b are next to each other, or two cs are next to each other, remove all two of them (i.e. delete ab, ba, cc). Determine the number of strings that can be reduced to c, the string of length 1, by the reducing processes mentioned above.
4	Let P be a point inside $\triangle ABC$. Let the perpendicular bisectors of PA, PB, PC be ℓ_1, ℓ_2, ℓ_3. Let $D = \ell_1 \cap \ell_2$, $E = \ell_2 \cap \ell_3$, $F = \ell_3 \cap \ell_1$. If A, B, C, D, E, F lie on a circle, prove that C, P, D are collinear.
–	day 2
5	For all positive real numbers a, b, c. Prove the folllowing inequality $$\frac{a}{c+5b} + \frac{b}{a+5c} + \frac{c}{b+5a} \geq \frac{1}{2}.$$
6	Let $T = \{1,2,...,10\}$. Find the number of bijective functions $f : T \to T$ that satis es the following for all $x \in T$: $f(f(x)) = x$ $\|f(x) - x\| \geq 2$
7	Let the incircle of $\triangle ABC$ meet BC, CA, AB at J, K, L. Let $D(\neq B, J), E(\neq C, K), F(\neq A, L)$ be points on BJ, CK, AL. If the incenter of $\triangle ABC$ is the circumcenter of $\triangle DEF$ and $\angle BAC = \angle DEF$, prove that $\triangle ABC$ and $\triangle DEF$ are isosceles triangles.
8	Prime p is called *Prime of the Year* if there exists a positive integer n such that $n^2 + 1 \equiv 0$ $(mod p^{2007})$. Prove that there are in nite number of *Primes of the Year*.