KJMO 2008

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–	day 1
1	In a $\triangle XYZ$, points A, B lie on segment ZX, C, D lie on segment XY, E, F lie on segment YZ. A, B, C, D lie on a circle, and $\frac{AZ \cdot EY \cdot ZB \cdot YF}{EZ \cdot CY \cdot ZF \cdot YD} = 1$. Let $L = ZX \cap DE$, $M = XY \cap AF$, $N = YZ \cap BC$. Prove that L, M, N are collinear.
2	Let $x, y \in \mathbb{R}$ such that $x > 2, y > 3$. Find the minimum value of $\frac{(x+y)^2}{\sqrt{x^2-4}+\sqrt{y^2-9}}$
3	For all positive integers n, prove that there are integers x, y relatively prime to 5 such that $x^2 + y^2 = 5^n$.
4	Let N be the set of positive integers. If $A, B, C \neq \emptyset, A \cap B = B \cap C = C \cap A = \emptyset$ and $A \cup B \cup C = N$, we say that A, B, C are partitions of N. Prove that there are no partitions of N, A, B, C, that satis fies the following. (i) $\forall a \in A, b \in B$, we have $a + b + 1 \in C$ (ii) $\forall b \in B, c \in C$, we have $b + c + 1 \in A$ (iii) $\forall c \in C, a \in A$, we have $c + a + 1 \in B$
–	day 2
5	Let there be a pentagon $ABCDE$ inscribed in a circle O. The tangent to O at E is parallel to AD. A point F lies on O and it is in the opposite side of A with respect to CD, and satisfi es $AB \cdot BC \cdot DF = AE \cdot ED \cdot CF$ and $\angle CFD = 2\angle BFE$. Prove that the tangent to O at B, E and line AF concur at one point.
6	If $d_1, d_2, ..., d_k$ are all distinct positive divisors of n, we defi ne $f_s(n) = d_1^s + d_2^s + .. + d_k^s$. For example, we have $f_1(3) = 1 + 3 = 4, f_2(4) = 1 + 2^2 + 4^2 = 21$. Prove that for all positive integers n, $n^3 f_1(n) - 2n f_9(n) + n^2 f_3(n)$ is divisible by 8.
7	Find all pairs of functions $f; g : R \to R$ such that for all reals $x.y \neq 0$: $$f(x+y) = g\left(\frac{1}{x} + \frac{1}{y}\right) \cdot (xy)^{2008}$$
8	There are 12 members in a club. The members created some small groups, which satisfy the following:

- The small group consists of 3 or 4 people.
- Also, for two arbitrary members, there exists exactly one small group that has both members.

Prove that all members are in the same number of small groups.