KJMO 2009

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–	day 1
1	For primes a, b, c that satis fies the following, calculate abc. $b+8$ is a multiple of a, and b^2-1 is a multiple of a and c. Also, $b+c=a^2-1$.
2	In an acute triangle $\triangle ABC$, let A', B', C' be the reflection of A, B, C with respect to BC, CA, AB. Let $D = B'C \cap BC'$, $E = CA' \cap C'A$, $F = A'B \cap AB'$. Prove that AD, BE, CF are concurrent
3	For two arbitrary reals x, y which are larger than 0 and less than 1. Prove that $$\frac{x^2}{x+y}+\frac{y^2}{1-x}+\frac{(1-x-y)^2}{1-y} \geq \frac{1}{2}.$$
4	There are n clubs composed of 4 students out of all 9 students. For two arbitrary clubs, there are no more than 2 students who are a member of both clubs. Prove that $n \leq 18$. Translator's Note. We can prove $n \leq 12$, and we can prove that the bound is tight. (Credits to rkm0959 for translation and document)
–	day 2
5	Acute triangle $\triangle ABC$ satis es $AB < AC$. Let the circumcircle of this triangle be O, and the mid-point of BC, CA, AB be D, E, F. Let P be the intersection of the circle with AB as its diameter and line DF, which is in the same side of C with respect to AB. Let Q be the intersection of the circle with AC as its diameter and the line DE, which is in the same side of B with respect to AC. Let $PQ \cap BC = R$, and let the line passing through R and perpendicular to BC meet AO at X. Prove that $AX = XR$.
6	If positive reals a, b, c, d satisfy $abcd = 1$. Prove the following inequality $$1 < \frac{b}{ab+b+1}+\frac{c}{bc+c+1}+\frac{d}{cd+d+1}+\frac{a}{da+a+1} < 2.$$
7	There are 3 students from Korea, China, and Japan, so total of 9 students are present. How many ways are there to make them sit down in a circular table, with equally spaced and equal chairs, such that the students from the same country do not sit next to each other? If array A can become array B by rotation, these two arrays are considered equal.

8 Let a, b, c, d, and e be positive integers. Are there any solutions to $a^2 + b^3 + c^5 + d^7 = e^{11}$?