KJMO 2010

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–	day 1
1	Prove that $7^{2^{20}} + 7^{2^{19}} + 1$ has at least 21 distinct prime divisors.
2	Let there be a $n \times n$ board. Write down 0 or 1 in all n^2 squares. For $1 \le k \le n$, let A_k be the product of all numbers in the kth row. How many ways are there to write down the numbers so that $A_1 + A_2 + ... + A_n$ is even?
3	In an acute triangle $\triangle ABC$, let there be point D on segment AC, E on segment AB such that $\angle ADE = \angle ABC$. Let the bisector of $\angle A$ hit BC at K. Let the foot of the perpendicular from K to DE be P, and the foot of the perpendicular from A to DE be L. Let Q be the midpoint of AL. If the incenter of $\triangle ABC$ lies on the circumcircle of $\triangle ADE$, prove that P, Q and the incenter of $\triangle ADE$ are collinear.
4	Let there be a sequence a_n such that $a_1 = 2, a_2 = 0, a_3 = 1, a_4 = 0$, and for $n \geq 1, a_{n+4}$ is the remainder when $a_n + 2a_{n+1} + 3a_{n+2} + 4a_{n+3}$ is divided by 9. Prove that there are no positive integer k such that $a_k = 0, a_{k+1} = 1, a_{k+2} = 0, a_{k+3} = 2$.
–	day 2
5	If reals x, y, z satises $tanx + tany + tanz = 2$ and $0 < x, y, z < \frac{\pi}{2}$. Prove that $$sin^2x + sin^2y + sin^2z < 1.$$
6	Let $n \in \mathbb{N}$ and p is the odd prime number. Define the sequence a_n such that $a_1 = pn + 1$ and $a_{k+1} = na_k + 1$ for all $k \in \mathbb{N}$. Prove that a_{p-1} is compound number.
7	Let $ABCD$ be a cyclic convex quadrilateral. Let E be the intersection of lines AB, CD. P is the intersection of line passing B and perpendicular to AC, and line passing C and perpendicular to BD. Q is the intersection of line passing D and perpendicular to AC, and line passing A and perpendicular to BD. Prove that three points E, P, Q are collinear.
8	In a rectangle with vertices $(0, 0), (0, 2), (n, 0), (n, 2)$, ($n$ is a positive integer) find the number of longest paths starting from $(0, 0)$ and arriving at $(n, 2)$ which satis fies the following:

At each movement, you can move right, up, left, down by 1. You cannot visit a point you visited before. You cannot move outside the rectangle.