KJMO 2011

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by parmenides51, seoneo

–	day 1
1	Real numbers a, b, c which are differ from 1 satisfies the following conditions; (1) $abc = 1$ (2) $a^2+b^2+c^2-\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=8(a+b+c)-8(ab+bc+ca)$ Find all possible values of expression $\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}$.
2	Let $ABCD$ be a cyclic quadrilateral inscirbed in circle O. Let the tangent to O at A meet BC at S, and the tangent to O at B meet CD at T. Circle with S as its center and passing A meets BC at E, and AE meets O again at $F(\neq A)$. The circle with T as its center and passing B meets CD at K. Let $P = BK \cap AC$. Prove that P, F, D are collinear if and only if $AB = AP$.
3	Let x, y be positive integers such that $gcd(x, y) = 1$ and $x + 3y^2$ is a perfect square. Prove that $x^2 + 9y^4$ can't be a perfect square.
4	For a positive integer n, ($n \geq 2$), find the number of sets with $2n+1$ points $P_0, P_1, ..., P_{2n}$ in the coordinate plane satisfying the following as its elements: - $P_0 = (0, 0), P_{2n} = (n, n)$ - For all $i = 1, 2, ..., 2n-1$, line P_iP_{i+1} is parallel to x-axis or y-axis and its length is 1. - Out of $2n$ lines$P_0P_1, P_1P_2, ..., P_{2n-1}P_{2n}$, there are exactly 4 lines that are enclosed in the domain $y \leq x$.
–	day 2
5	In triangle ABC, ($AB \neq AC$), let the orthocenter be H, circumcenter be O, and the midpoint of BC be M. Let $HM \cap AO = D$. Let P, Q, R, S be the midpoints of AB, CD, AC, BD. Let $X = PQ \cap RS$. Find AH/OX.
6	For a positive integer n, define the set S_n as $S_n = \{(a, b) \mid a, b \in N, lcm[a, b] = n\}$. Let $f(n)$ be the sum of $\phi(a)\phi(b)$ for all $(a, b) \in S_n$. If a prime p relatively prime to n is a divisor of $f(n)$, prove that there exists a prime $q \mid n$ such that $p \mid q^2-1$.
7	For those real numbers $x_1, x_2, \ldots, x_{2011}$ where each of which satisfies $0 \leq x_1 \leq 1$ ($i = 1, 2, \ldots, 2011$),

find the maximum of

$$x_1^3 + x_2^3 + \cdots + x_{2011}^3 - (x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2)$$

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8 There are n students each having r positive integers. Their nr positive integers are all different. Prove that we can divide the students into k classes satisfying the following conditions:
(a) $k \leq 4r$
(b) If a student A has the number m, then the student B in the same class can't have a number ℓ such that $(m-1)! < \ell < (m+1)! + 1$
