KJMO 2012

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– day 1

1 Prove the following inequality where positive reals a, b, c satisfies $ab+bc+ca=1$.

$$\frac{a+b}{\sqrt{ab(1-ab)}}+\frac{b+c}{\sqrt{bc(1-bc)}}+\frac{c+a}{\sqrt{ca(1-ca)}}\le\frac{\sqrt{2}}{abc}$$

2 A pentagon $ABCDE$ is inscribed in a circle O, and satis fies $\angle A = 90^o, AB = CD$. Let F be a point on segment AE. Let BF hit O again at $J(\neq B), CE \cap DJ = K, BD \cap FK = L$. Prove that B, L, E, F are cyclic.

3 Find all $l, m, n \in \mathbb{N}$ that satisfies the equation $5^l 43^m + 1 = n^3$

4 There are n students $A_1, A_2, ..., A_n$ and some of them shaked hands with each other. (A_i and $A-j$ can shake hands more than one time.) Let the student A_i shaked hands d_i times. Suppose $d_1 + d_2 + ... + d_n > 0$. Prove that there exist $1 \le i < j \le n$ satisfying the following conditions:
(a) Two students A_i and A_j shaked hands each other.
(b) $\frac{(d_1+d_2+...+d_n)^2}{n^2} \le d_i d_j$

– day 2

5 Let $ABCD$ be a cyclic quadrilateral inscribed in a circle O ($AB > AD$), and let E be a point on segment AB such that $AE = AD$. Let $AC \cap DE = F$, and $DE \cap O = K(\neq D)$. The tangent to the circle passing through C, F, E at E hits AK at L. Prove that $AL = AD$ if and only if $\angle KCE = \angle ALE$.

6 $p > 3$ is a prime number such that $p|2^{p-1} - 1$ and $p \nmid 2^x - 1$ for $x = 1, 2, ..., p-2$. Let $p = 2k+3$. Now we define sequence $\{a_n\}$ as

$$a_i = a_{i+k} = 2^i \ (1 \le i \le k), \quad a_{j+2k} = a_j a_{j+k} \ (j \le 1)$$

Prove that there exist $2k$ consecutive terms of sequence $a_{x+1}, a_{x+2}, ..., a_{x+2k}$ such that $a_{x+i} \not\equiv a_{x+j}$ (mod p) for all $1 \le i < j \le 2k$.

7 If all x_k $(k = 1, 2, 3, 4, 5)$ are positive reals, and $\{a_1, a_2, a_3, a_4, a_5\} = \{1, 2, 3, 4, 5\}$, find the maximum of

$$\frac{(\sqrt{s_1x_1}+\sqrt{s_2x_2}+\sqrt{s_3x_3}+\sqrt{s_4x_4}+\sqrt{s_5x_5})^2}{a_1x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5}$$

$(s_k = a_1 + a_2 + ... + a_k)$

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8 Let there be n students, numbered 1 through n. Let there be n cards with numbers 1 through n written on them. Each student picks a card from the stack, and two students are called a pair if they pick each other's number. Let the probability that there are no pairs be p_n.
Prove that $p_n - p_{n-1}$ is 0 is n is odd, and
prove that $p_n - p_{n-1} = \frac{1}{(-2)^k k^{1-k}}$ if $n = 2k$.
