KJMO 2013

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by parmenides51, seoneo, syk0526

–	day 1
1	Compare the magnitude of the following three numbers. $$\sqrt[3]{\frac{25}{3}},\ \sqrt[3]{\frac{1148}{135}},\ \frac{\sqrt[3]{25}}{3}+\sqrt[3]{\frac{6}{5}}$$
2	A pentagon $ABCDE$ is inscribed in a circle O, and satis es $AB = BC, AE = DE$. The circle that is tangent to DE at E and passing A hits EC at F and BF at $G(\neq F)$. Let $DG \cap O = H(\neq D)$. Prove that the tangent to O at E is perpendicular to HA.
3	$\{a_n\}$ is a positive integer sequence such that $a_{i+2} = a_{i+1} + a_i$ (for all $i \geq 1$). For positive integer n, de fine as $$b_n = \frac{1}{a_{2n+1}}\Sigma_{i=1}^{4n-2} a_i$$ Prove that b_n is positive integer.
4	Prove that there exists a prime number p such that the minimum positive integer n such that $p\vert 2^n - 1$ is 3^{2013}.
–	day 2
5	In an acute triangle $\triangle ABC, \angle A > \angle B$. Let the midpoint of AB be D, and let the foot of the perpendicular from A to BC be E, and B from CA be F. Let the circumcenter of $\triangle DEF$ be O. A point J on segment BE satisfi es $\angle ODC = \angle EAJ$. Prove that $AJ \cap DC$ lies on the circumcircle of $\triangle BDE$.
6	Find all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $$f(mn) = \operatorname{lcm}(m, n) \cdot \gcd(f(m), f(n))$$ for all positive integer m, n.
7	Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be such that for every positive integer n, followings are satisfied. i. $f(n + 1) > f(n)$ ii. $f(f(n)) = 2n + 2$

Find the value of $f(2013)$.
(Here, $\mathbb{N}$ is the set of all positive integers.)

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8 Drawing all diagonals in a regular 2013-gon, the regular 2013-gon is divided into non-overlapping polygons.
Prove that there exist exactly one 2013-gon out of all such polygons.
