KJMO 2014

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–	day 1
1	Given $\triangle ABC$ with incenter I. Line AI meets BC at D. The incenter of $\triangle ABD, \triangle ADC$ are E, F, respectively. Line DE meets the circumcircle of $\triangle BCE$ at$P(\neq E)$ and line DF meets the circumcircle of $\triangle BCF$ at$Q(\neq F)$. Show that the midpoint of BC lies on the circumcircle of $\triangle DPQ$.
2	Let there be $2n$ positive reals $a_1, a_2, ..., a_{2n}$. Let $s = a_1 + a_3 + ... + a_{2n-1}$, $t = a_2 + a_4 + ... + a_{2n}$, and $x_k = a_k + a_{k+1} + ... + a_{k+n-1}$ (indices are taken modulo $2n$). Prove that $$\frac{s}{x_1} + \frac{t}{x_2} + \frac{s}{x_3} + \frac{t}{x_4} + ... + \frac{s}{x_{2n-1}} + \frac{t}{x_{2n}} > \frac{2n^2}{n+1}$$
3	Find the number of n-movement on the following graph, starting from S.
4	Positive integers p, q, r satisfy $gcd(a, b, c) = 1$. Prove that there exists an integer a such that $gcd(p, q + ar) = 1$.
–	day 2
5	For positive integers x, y, find all pairs (x, y) such that $x^2y + x$ is a multiple of $xy^2 + 7$.
6	Let $p = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}$. For nonnegative reals x, y, z satisfying $(x-1)^2+(y-1)^2+(z-1)^2 = 27$, find the maximum value of $x^p + y^p + z^p$.
7	In a parallelogram $\square ABCD$ $(AB < BC)$ The incircle of $\triangle ABC$ meets $\overline{BC}$ and $\overline{CA}$ at P, Q. The incircle of $\triangle ACD$ and $\overline{CD}$ meets at R. Let S = $PQ \cap AD$ U = $AR \cap CS$ T, a point on $\overline{BC}$ such that $\overline{AB} = \overline{BT}$ Prove that AT, BU, PQ are concurrent
8	Let there be n students and m clubs. The students joined the clubs so that the following is true: - For all students x, you can choose some clubs such that x is the only student who joined all of the chosen clubs.

Let the number of clubs each student joined be $a_1, a_2, ..., a_m$. Prove that

$$a_1!(m-a_1)! + a_2!(m-a_2)! + ... + a_n!(m-a_n)! \leq m!$$