KJMO 2015

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–	day 1
1	In an acute, scalene triangle $\triangle ABC$, let O be the circumcenter. Let M be the midpoint of AC. Let the perpendicular from A to BC be D. Let the circumcircle of $\triangle OAM$ hit DM at $P(\neq M)$. Prove that B, O, P are colinear.
2	For a positive integer m, prove that the number of pairs of positive integers (x, y) which satisfies the following two conditions is even or 0. (i): $x^2 - 3y^2 + 2 = 16m$ (ii): $2y \leq x - 1$
3	For all nonnegative integer i, there are seven cards with 2^i written on it. How many ways are there to select the cards so that the numbers add up to n?
4	Reals a, b, c, x, y satisfy $a^2 + b^2 + c^2 = x^2 + y^2 = 1$. Find the maximum value of $$(ax + by)^2 + (bx + cy)^2$$
–	day 2
5	Let I be the incenter of an acute triangle $\triangle ABC$, and let the incircle be Γ. Let the circumcircle of $\triangle IBC$ hit Γ at D, E, where D is closer to B and E is closer to C. Let $\Gamma \cap BE = K(\neq E)$, $CD \cap BI = T$, and $CD \cap \Gamma = L(\neq D)$. Let the line passing T and perpendicular to BI meet Γ at P, where P is inside $\triangle IBC$. Prove that the tangent to Γ at P, KL, BI are concurrent.
6	Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that (i): For different reals x, y, $f(x) \neq f(y)$. (ii): For all reals x, y, $f(x + f(f(-y))) = f(x) + f(f(y))$
7	For a polynomial $f(x)$ with integer coefficients and degree no less than 1, prove that there are infinitely many primes p which satisfies the following. There exists an integer n such that $f(n) \neq 0$ and $\|f(n)\|$ is a multiple of p.

8 A positive integer n is given. If there exist sets $F_1, F_2, \cdots F_m$ satisfying the following, prove that $m \leq n$.
(For sets A, B, $|A|$ is the number of elements in A. $A - B$ is the set of elements that are in A but not B)

(i): For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots n\}$

(ii): $|F_1| \leq |F_2| \leq \cdots \leq |F_m|$

(iii): For all $1 \leq i < j \leq m$, $|F_i - F_j| = 1$.