

**Saint Petersburg Mathematical Olympiad 2019**

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– Grade 11

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- 1 A polynomial  $f(x)$  of degree 2000 is given. It's known that  $f(x^2 - 1)$  has exactly 3400 real roots while  $f(1 - x^2)$  has exactly 2700 real roots. Prove that there exist two real roots of  $f(x)$  such that the difference between them is less than 0.002.

(.)

Thanks to the user Vlados021 for translating the problem.

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- 2 On the blackboard there are written 100 different positive integers. To each of these numbers is added the gcd of the 99 other numbers. In the new 100 numbers, is it possible for 3 of them to be equal.

(.)

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- 3 Kid and Karlsson play a game. Initially they have a square piece of chocolate  $2019 \times 2019$  grid with  $1 \times 1$  cells. On every turn Kid divides an arbitrary piece of chocolate into three rectangular pieces by cells, and then Karlsson chooses one of them and eats it. The game finishes when it's impossible to make a legal move. Kid wins if there was made an even number of moves, Karlsson wins if there was made an odd number of moves.

Who has the winning strategy?

(.)

Thanks to the user Vlados021 for translating the problem.

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- 4 A non-equilateral triangle  $\triangle ABC$  of perimeter 12 is inscribed in circle  $\omega$ . Points  $P$  and  $Q$  are arc midpoints of arcs  $ABC$  and  $ACB$ , respectively. Tangent to  $\omega$  at  $A$  intersects line  $PQ$  at  $R$ . It turns out that the midpoint of segment  $AR$  lies on line  $BC$ . Find the length of the segment  $BC$ .

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- 5 Baron Munchhausen has a collection of stones, such that they are of 1000 distinct whole weights,  $2^{1000}$  stones of every weight. Baron states that if one takes exactly one stone of every weight, then the weight of all these 1000 stones chosen will be less than  $2^{1010}$ , and there is no other way to obtain this weight by picking another set of stones of the collection.

Can this statement happen to be true?

(.)

Thanks to the user Vlados021 for translating the problem.

- 6** Suppose that there are roads  $AB$  and  $CD$  but there are no roads  $BC$  and  $AD$  between four cities  $A, B, C,$  and  $D$ . Define *restructuring* to be the changing a pair of roads  $AB$  and  $CD$  to the pair of roads  $BC$  and  $AD$ . Initially there were some cities in a country, some of which were connected by roads and for every city there were exactly 100 roads starting in it. The minister drew a new scheme of roads, where for every city there were also exactly 100 roads starting in it. It's known also that in both schemes there were no cities connected by more than one road. Prove that it's possible to obtain the new scheme from the initial after making a finite number of restructurings.

(.)

Thanks to the user Vlados021 for translating the problem.

- 7** Let  $\omega$  and  $O$  be respectively the circumcircle and the circumcenter of a triangle  $ABC$ . The line  $AO$  intersects  $\omega$  second time at  $A'$ .  $M_B$  and  $M_C$  are the midpoints of  $AC$  and  $AB$ , respectively. The lines  $A'M_B$  and  $A'M_C$  intersect  $\omega$  secondly at points  $B'$  and  $C'$ , and also intersect  $BC$  at points  $D_B$  and  $D_C$ , respectively. The circumcircles of  $CD_B B'$  and  $BD_C C'$  intersect at points  $P$  and  $Q$ . Prove that  $O, P, Q$  are collinear.

(.)

Thanks to the user Vlados021 for translating the problem.

– Grade 10

- 1** For a non-constant arithmetic progression  $(a_n)$  there exists a natural  $n$  such that  $a_n + a_{n+1} = a_1 + \dots + a_{3n-1}$ . Prove that there are no zero terms in this progression.
- 2** Every two of the  $n$  cities of Ruritania are connected by a direct flight of one from two airlines. Promonopoly Committee wants at least  $k$  flights performed by one company. To do this, he can at least every day to choose any three cities and change the ownership of the three flights connecting these cities each other (that is, to take each of these flights from a company that performs it, and pass the other). What is the largest  $k$  committee knowingly will be able to achieve its goal in no time, no matter how the flights are distributed hour?

- 3** Let  $a, b$  and  $c$  be non-zero natural numbers such that  $c \geq b$ . Show that

$$a^b (a + b)^c > c^b a^c.$$

4 Given a convex quadrilateral  $ABCD$ . The medians of the triangle  $ABC$  intersect at point  $M$ , and the medians of the triangle  $ACD$  at point  $N$ . The circle, circumscribed around the triangle  $ACM$ , intersects the segment  $BD$  at the point  $K$  lying inside the triangle  $AMB$ . It is known that  $\angle MAN = \angle ANC = 90^\circ$ . Prove that  $\angle AKD = \angle MKC$ .

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5 A class has 25 students. The teacher wants to stock  $N$  candies, hold the Olympics and give away all  $N$  candies for success in it (those who solve equally tasks should get equally, those who solve less get less, including, possibly, zero candies). At what smallest  $N$  this will be possible, regardless of the number of tasks on Olympiad and the student successes?

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6 Is it possible to arrange everything in all cells of an infinite checkered plane all natural numbers (once) so that for each  $n$  in each square  $n \times n$  the sum of the numbers is a multiple of  $n$ ?

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7 In a square  $10^{2019} \times 10^{2019}$ ,  $10^{4038}$  points are marked. Prove that there is such a rectangle with sides parallel to the sides of a square whose area differs from the number of points located in it by at least 6.

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– Grade 9

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1 A natural number is called a palindrome if it is read in the same way, from left to right and from right to left (in particular, the last digit of the palindrome coincides with the first and therefore not equal to zero). Squares of two different natural numbers have 1001 digits. Prove that strictly between these squares, there is one palindrome.

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2 In the city built are 2019 metro stations. Some pairs of stations are connected, tunnels, and from any station through the tunnels you can reach any other. The mayor ordered to organize several metro lines: each line should include several different stations connected in series by tunnels (several lines can pass through the same tunnel), and in each station must lie at least on one line. To save money no more than  $k$  lines should be made. It turned out that the order of the mayor is not feasible. What is the largest  $k$  it could to happen?

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3 Prove that the distance between the midpoint of side  $BC$  of triangle  $ABC$  and the midpoint of the arc  $ABC$  of its circumscribed circle is not less than  $AB/2$

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4 Olya wrote fractions of the form  $1/n$  on cards, where  $n$  is all possible divisors the numbers  $6^{100}$  (including the unit and the number itself). These cards she laid out in some order. After that, she wrote down the number on the first card, then the sum of the numbers on the first and second cards, then the sum of the numbers on the first three cards, etc., finally, the sum of the numbers on all the cards. Every amount Olya recorded on the board in the form of irreducible fraction. What is the least different denominators could be on the numbers on the board?

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5 Call the *improvement* of a positive number its replacement by a power of two. (i.e. one of the numbers 1, 2, 4, 8, ...), for which it increases, but not more than than 3 times. Given  $2^{100}$  positive

numbers with a sum of  $2^{100}$ . Prove that you can erase some of them, and *improve* each of the other numbers so that the sum the resulting numbers were again  $2^{100}$ .

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- 6** The bisectors  $BB_1$  and  $CC_1$  of the acute triangle  $ABC$  intersect in point  $I$ . On the extensions of the segments  $BB_1$  and  $CC_1$ , the points  $B'$  and  $C'$  are marked, respectively. So, the quadrilateral  $AB'IC'$  is a parallelogram. Prove that if  $\angle BAC = 60^\circ$ , then the straight line  $B'C'$  passes through the intersection point of the circumscribed circles of the triangles  $BC_1B'$  and  $CB_1C'$ .
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- 7** In a circle there are 2019 plates, on each lies one cake. Petya and Vasya are playing a game. In one move, Petya points at a cake and calls number from 1 to 16, and Vasya moves the specified cake to the specified number of check clockwise or counterclockwise (Vasya chooses the direction each time). Petya wants at least some  $k$  pastries to accumulate on one of the plates and Vasya wants to stop him. What is the largest  $k$  Petya can succeed?
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