Saint Petersburg Mathematical Olympiad 2016

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–	grade 11

1   In the sequence of integers (a_n), the sum $a_m + a_n$ is divided by $m + n$ with any different m and n. Prove that a_n is a multiple of n for any n.

2   The rook, standing on the surface of the checkered cube, beats the cells, located in the same row as well as on the
continuations of this series through one or even several edges. (The picture shows an example for a $4 \times 4 \times 4$ cube,visible cells that some beat the rook, shaded gray.) What is the largest number do not beat each other rooks can be placed on the surface of the cube $50 \times 50 \times 50$?

3   In a tetrahedron, the midpoints of all the edges lie on the same sphere. Prove that it's altitudes intersect at one point.

4 $N > 4$ points move around the circle, each with a constant speed. For Any four of them have a moment in time when they all meet. Prove that is the moment when all the points meet.

5   Incircle of $\triangle ABC$ touch AC at D. BD intersect incircle at E. Points F, G on incircle are such points, that $FE \parallel BC, GE \parallel AB$. I_1, I_2 are incenters of DEF, DEG.
Prove that $I_1 I_2 \perp$ bisector of $\angle ABC$

7   A polynomial P with real coefficients is called *great,* if for some integer $a > 1$ and for all integers x, there exists an integer z such that $aP(x) = P(z)$. Find all *great* polynomials.

Proposed by A. Golovanov

–	grade 10

1   Sasha multiplied all the divisors of the natural number n. Fedya increased each divider by 1, and then multiplied the results. If the product found Fedya is divided by the product found by Sasha , what can n be equal to ?

2   Given the positive numbers $x_1, x_2, ..., x_n$, such that $x_i \leq 2x_j$ with $1 \leq i < j \leq n$.
Prove that there are positive numbers $y_1 \leq y_2 \leq ... \leq y_n$, such that $x_k \leq y_k \leq 2x_k$ for all $k = 1, 2, ..., n$

3   The circle inscribed in the triangle ABC is tangent to side AC at point B_1, and to side BC at point A_1. On the side AB there is a point K such that $AK = KB_1, BK = KA_1$. Prove that

$\angle ACB \geq 60$

4 The cells of a square 100×100 table are colored in one of two colors, black or white. A coloring is called admissible if for any row or column, the number b of black colored cells satisfies $50 \leq b \leq 60$. It is permitted to recolor a cell if by doing so the resulting configuration is still admissible. Prove that one can transition from any admissible coloring of the board to any other using a sequence of valid recoloring operations.

5 Points A and P are marked in the plane not lying on the line ℓ. For all right triangles ABC with hypotenuse on ℓ, show that the circumcircle of triangle BPC passes through a fixed point other than P.

6 The circle contains a closed 100-part broken line, such that no three segments pass through one point. All its corners are obtuse, and their sum in degrees is divided by 720. Prove that this broken line has an odd number of self-intersection points.

7 A polynomial $P(x)$ with integer coefficients and a positive integer $a > 1$, are such that for all integers x, there exists an integer z such that $aP(x) = P(z)$. Find all such pairs of $(P(x), a)$.

– grade 9

1 Given three quadratic trinomials f, g, h without roots. Their elder coefficients are the same, and all their coefficients for x are different. Prove that there is a number c such that the equations $f(x) + cg(x) = 0$ and $f(x) + ch(x) = 0$ have a common root.

2 On a 300×300 board, several rooks are placed that beat the entire board. Within this case, each rook beats no more than one other rook. At what least k, it is possible to state that there is at least one rook in each $k \times k$ square ?

3 On the side AB of the non-isosceles triangle ABC, let the points P and Q be so that $AC = AP$ and $BC = BQ$. The perpendicular bisector of the segment PQ intersects the angle bisector of the $\angle C$ at the point R (inside the triangle). Prove that $\angle ACB + \angle PRQ = 180^o$.

4 Two different prime numbers p and q differ in less than 2 times. Prove that exists two consecutive natural numbers, such that largest prime divisor of first number is p, and largest prime divisor of second number is q.

5 Kostya and Sergey play a game on a white strip of length 2016 cells. Kostya (he plays first) in one move should paint black over two neighboring white cells. Sergey should paint either one white cell either three neighboring white cells. It is forbidden to make a move, after which a white cell is formed the doesn't having any white neighbors. Loses the one that can make no other move. However, if all cells are painted, then Kostya wins. Who will win if he plays the right game (has a winning strategy)?

6   Incircle of $\triangle ABC$ touch AC at D. BD intersect incircle at E. Points F, G on incircle are such points, that $FE \parallel BC, GE \parallel AB$. I_1, I_2 are incenters of DEF, DEG. Prove that angle bisector of $\angle GDF$ passes though the midpoint of $I_1 I_2$.

7   A sequence of N consecutive positive integers is called *good* if it is possible to choose two of these numbers so that their product is divisible by the sum of the other $N - 2$ numbers. For which N do there exist infinitely many *good* sequences?