All-Russian Olympiad 2019

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– grade 9

1 There are 5 points on plane. Prove that you can chose some of them and shift them such that distances between shifted points won't change and as a result there will be symetric by some line set of 5 points.

2 Find minimal natural n for which there exist integers $a_1, a_2, \dots, a_n$ such that quadratic trinom

$$x^2 - 2(a_1 + a_2 + \cdots + a_n)^2 x + (a_1^4 + a_2^4 + \cdots + a_n^4 + 1)$$

has at least one integral root.

3 Circle Ω with center O is the circumcircle of an acute triangle $\triangle ABC$ with $AB < BC$ and orthocenter H.
On the line BO there is point D such that O is between B and D and $\angle ADC = \angle ABC$. The semi-line starting at H and parallel to BO wich intersects segment AC , intersects Ω at E. Prove that $BH = DE$.

4 10000 children came to a camp; every of them is friend of exactly eleven other children in the camp (friendship is mutual). Every child wears T-shirt of one of seven rainbow's colours; every two friends' colours are different. Leaders demanded that some children (at least one) wear T-shirts of other colours (from those seven colours). Survey pointed that 100 children didn't want to change their colours [translator's comment: it means that any of these 100 children (and only them) can't change his (her) colour such that still every two friends' colours will be different]. Prove that some of other children can change colours of their T-shirts such that as before every two friends' colours will be different.

5 In kindergarten, nurse took $n > 1$ identical cardboard rectangles and distributed them to n children; every child got one rectangle. Every child cut his (her) rectangle into several identical squares (squares of different children could be different). Finally, the total number of squares was prime. Prove that initial rectangles was squares.

6 There is point D on edge AC isosceles triangle ABC with base BC. There is point K on the smallest arc CD of circumcircle of triangle BCD. Ray CK intersects line parallel to line BC through A at point T. Let M be midpoint of segment DT. Prove that $\angle AKT = \angle CAM$.

7 Among 16 coins there are 8 heavy coins with weight of 11 g, and 8 light coins with weight of 10 g, but it's unknown what weight of any coin is. One of the coins is anniversary. How to know, is anniversary coin heavy or light, via three weighings on scales with two cups and without any weight?

8 For a, b, c be real numbers greater than 1, prove that

$$\frac{a+b+c}{4} \geq \frac{\sqrt{ab-1}}{b+c} + \frac{\sqrt{bc-1}}{c+a} + \frac{\sqrt{ca-1}}{a+b}.$$

– grade 10

1 Each point A in the plane is assigned a real number $f(A)$. It is known that $f(M) = f(A) + f(B) + f(C)$, whenever M is the centroid of $\triangle ABC$. Prove that $f(A) = 0$ for all points A.

2 Pasha and Vova play the following game, making moves in turn; Pasha moves first. Initially, they have a large piece of plasticine. By a move, Pasha cuts one of the existing pieces into three(of arbitrary sizes), and Vova merges two existing pieces into one. Pasha wins if at some point there appear to be 100 pieces of equal weights. Can Vova prevent Pasha's win?

3 An interstellar hotel has 100 rooms with capacities $101, 102, \ldots, 200$ people. These rooms are occupied by n people in total. Now a VIP guest is about to arrive and the owner wants to provide him with a personal room. On that purpose, the owner wants to choose two rooms A and B and move all guests from A to B without exceeding its capacity. Determine the largest n for which the owner can be sure that he can achieve his goal no matter what the initial distribution of the guests is.

4 Let ABC be an acute-angled triangle with $AC < BC$. A circle passes through A and B and crosses the segments AC and BC again at A_1 and B_1 respectively. The circumcircles of A_1B_1C and ABC meet each other at points P and C. The segments AB_1 and A_1B intersect at S. Let Q and R be the reflections of S in the lines CA and CB respectively. Prove that the points P, Q, R, and C are concyclic.

5 In a kindergarten, a nurse took n congruent cardboard rectangles and gave them to n kids, one per each. Each kid has cut its rectangle into congruent squares(the squares of different kids could be of different sizes). It turned out that the total number of the obtained squares is a prime number. Prove that all the initial squares were in fact squares.

6 Let L be the foot of the internal bisector of $\angle B$ in an acute-angled triangle ABC. The points D and E are the midpoints of the smaller arcs AB and BC respectively in the circumcircle ω of $\triangle ABC$. Points P and Q are marked on the extensions of the segments BD and BE beyond D and E respectively so that $\measuredangle APB = \measuredangle CQB = 90^\circ$. Prove that the midpoint of BL lies on the line PQ.

7 24 students attend a mathematical circle. For any team consisting of 6 students, the teacher considers it to be either **GOOD** or **OK**. For the tournament of mathematical battles, the teacher wants to partition all the students into 4 teams of 6 students each. May it happen that every such partition contains either 3 **GOOD** teams or exactly one **GOOD** team and both options are present?

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8 Let $P(x)$ be a non-constant polynomial with integer coefficients and let n be a positive integer. The sequence $a_0, a_1, \ldots$ is defined as follows: $a_0 = n$ and $a_k = P(a_{k-1})$ for all positive integers k. Assume that for every positive integer b the sequence contains a bth power of an integer greater than 1. Show that $P(x)$ is linear.

– grade 11

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1 There is located real number $f(A)$ in any point A on the plane. It's known that if M will be centroid of triangle ABC then $f(M) = f(A) + f(B) + f(C)$. Prove that $f(A) = 0$ for all points A.

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2 Is it true, that for all pairs of non-negative integers a and b , the system

$$\tan 13x \tan ay = 1$$
$$\tan 21x \tan by = 1$$

has at least one solution?

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3 We are given n coins of different weights and n balances, $n > 2$. On each turn one can choose one balance, put one coin on the right pan and one on the left pan, and then delete these coins out of the balance. It's known that one balance is wrong (but it's not known ehich exactly), and it shows an arbitrary result on every turn. What is the smallest number of turns required to find the heaviest coin?

Thanks to the user Vlados021 for translating the problem.

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4 A triangular pyramid $ABCD$ is given. A sphere ω_A is tangent to the face BCD and to the planes of other faces in points don't lying on faces. Similarly, sphere ω_B is tangent to the face ACD and to the planes of other faces in points don't lying on faces. Let K be the point where ω_A is tangent to ACD, and let L be the point where ω_B is tangent to BCD. The points X and Y are chosen on the prolongations of AK and BL over K and L such that $\angle CKD = \angle CXD + \angle CBD$ and $\angle CLD = \angle CYD + \angle CAD$. Prove that the distances from the points X, Y to the midpoint of CD are the same.

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5 Radii of five concentric circles $\omega_0, \omega_1, \omega_2, \omega_3, \omega_4$ form a geometric progression with common ratio q in this order. What is the maximal value of q for which it's possible to draw a broken line $A_0A_1A_2A_3A_4$ consisting of four equal segments such that A_i lies on ω_i for every $i = \overline{0, 4}$?

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6 In the segment AC of an isosceles triangle $\triangle ABC$ with base BC is chosen a point D. On the smaller arc CD of the circumcircle of $\triangle BCD$ is chosen a point K. Line CK intersects the line through A parallel to BC at T. M is the midpoint of segment DT. Prove that $\angle AKT = \angle CAM$.

(A.Kuznetsov)

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7 There are non-constant polynom $P(x)$ with integral coefficients and natural number n. Suppose that $a_0 = n$, $a_k = P(a_{k-1})$ for any natural k. Finally, for every natural b there is number in sequence $a_0, a_1, a_2, \dots$ that is b-th power of some natural number that is more than 1. Prove that $P(x)$ is linear polynom.

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8 A positive integer n is given. A cube $3 \times 3 \times 3$ is built from 26 white and 1 black cubes $1 \times 1 \times 1$ such that the black cube is in the center of $3 \times 3 \times 3$-cube. A cube $3n \times 3n \times 3n$ is formed by n^3 such $3 \times 3 \times 3$-cubes. What is the smallest number of white cubes which should be colored in red in such a way that every white cube will have at least one common vertex with a red one.

Thanks to the user Vlados021 for translating the problem.
