China Second Round Olympiad 2018

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– (A)

1 Let $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n, A, B$ are positive reals such that $a_i \le b_i, a_i \le A\,(i = 1, 2, \cdots, n)$ and $\frac{b_1 b_2 \cdots b_n}{a_1 a_2 \cdots a_n} \le \frac{B}{A}$. Prove that

$$\frac{(b_1+1)(b_2+1)\cdots(b_n+1)}{(a_1+1)(a_2+1)\cdots(a_n+1)} \le \frac{B+1}{A+1}.$$

2 In triangle $\triangle ABC$, $AB < AC$, M, D, E are the midpoints of BC, the arcs BAC and BC of the circumcircle of $\triangle ABC$ respectively. The incircle of $\triangle ABC$ touches AB at F, AE meets BC at G, and the perpendicular to AB at B meets segment EF at N. If $BN = EM$, prove that DF is perpendicular to FG.

3 Let n, k, m be positive integers, where $k \ge 2$ and $n \le m < \frac{2k-1}{k}n$. Let A be a subset of $\{1, 2, \ldots, m\}$ with n elements. Prove that every integer in the range $\left(0, \frac{n}{k-1}\right)$ can be expressed as $a - b$, where $a, b \in A$.

4 Define sequence $\{a_n\}$: a_1 is any positive integer, and for any positive integer $n \ge 1$, a_{n+1} is the smallest positive integer coprime to $\sum_{i=1}^{n} a_i$ and not equal to $a_1, \ldots, a_n$. Prove that every positive integer appears in the sequence $\{a_n\}$.

– (B)

1 Let $a, b \in \mathbb{R}, f(x) = ax + b + \frac{9}{x}$. Prove that there exists $x_0 \in [1, 9]$, such that $|f(x_0)| \ge 2$.

2 In triangle $\triangle ABC$, $AB = AC$. Let D be on segment AC and E be a point on the extended line BC such that C is located between B and E and $\frac{AD}{DC} = \frac{BC}{2CE}$. Let ω be the circle with diameter AB, and ω intersects segment DE at F. Prove that B, C, F, D are concyclic.

3 Let set $A = \{1, 2, \ldots, n\}$, and X, Y be two subsets (not necessarily distinct) of A. Define that max$X$ and min$Y$ represent the greatest element of X and the least element of Y, respectively. Determine the number of two-tuples (X, Y) which satisfies max$X >$ min$Y$.

4 Prove that for any integer $a \ge 2$ and positive integer n, there exist positive integer k such that $a^k + 1, a^k + 2, \ldots, a^k + n$ are all composite numbers.