

**Balkan MO 2019**

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**1** Let  $\mathbb{P}$  be the set of all prime numbers. Find all functions  $f : \mathbb{P} \rightarrow \mathbb{P}$  such that:

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all  $p, q \in \mathbb{P}$ .

*Proposed by Dorlir Ahmeti, Albania*

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**2** Let  $a, b, c$  be real numbers such that  $0 \leq a \leq b \leq c$  and  $a + b + c = ab + bc + ca > 0$ . Prove that  $\sqrt{bc}(a + 1) \geq 2$  and determine the equality cases.

(Edit: Proposed by sir Leonard Giugiuc, Romania)

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**3** Let  $ABC$  be an acute scalene triangle. Let  $X$  and  $Y$  be two distinct interior points of the segment  $BC$  such that  $\angle CAX = \angle YAB$ . Suppose that: 1)  $K$  and  $S$  are the feet of the perpendiculars from  $B$  to the lines  $AX$  and  $AY$  respectively. 2)  $T$  and  $L$  are the feet of the perpendiculars from  $C$  to the lines  $AX$  and  $AY$  respectively. Prove that  $KL$  and  $ST$  intersect on the line  $BC$ .

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**4** A grid consists of all points of the form  $(m, n)$  where  $m$  and  $n$  are integers with  $|m| \leq 2019$ ,  $|n| \leq 2019$  and  $|m| + |n| < 4038$ . We call the points  $(m, n)$  of the grid with either  $|m| = 2019$  or  $|n| = 2019$  the *boundary points*. The four lines  $x = \pm 2019$  and  $y = \pm 2019$  are called *boundary lines*. Two points in the grid are called *neighbours* if the distance between them is equal to 1.

Anna and Bob play a game on this grid.

Anna starts with a token at the point  $(0, 0)$ . They take turns, with Bob playing first.

1) On each of his turns. Bob *deletes* at most two boundary points on each boundary line.

2) On each of her turns. Anna makes exactly three *steps*, where a *step* consists of moving her token from its current point to any neighbouring point, which has not been deleted.

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins.

Does Anna have a winning strategy?

*Proposed by Demetres Christofides, Cyprus*

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