KJMO 2017

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by parmenides51, Starrysky, rkm0959, enneper

– day 1

1 Find all positive integer n and nonnegative integer $a_1, a_2, \dots, a_n$ satisfying:

i divides exactly a_i numbers among $a_1, a_2, \dots, a_n$, for each $i = 1, 2, \dots, n$.
(0 is divisible by all integers.)

2 Let there be a scalene triangle ABC, and its incircle hits BC, CA, AB at D, E, F. The perpendicular bisector of BC meets the circumcircle of ABC at P, Q, where P is on the same side with A with respect to BC. Let the line parallel to AQ and passing through D meet EF at R. Prove that the intersection between EF and PQ lies on the circumcircle of BCR.

3 Find all $n > 1$ and integers $a_1, a_2, \dots, a_n$ satisfying the following three conditions:
(i) $2 < a_1 \le a_2 \le \dots \le a_n$
(ii) $a_1, a_2, \dots, a_n$ are divisors of $15^{25} + 1$.
(iii) $2 - \frac{2}{15^{25}+1} = \left(1 - \frac{2}{a_1}\right) + \left(1 - \frac{2}{a_2}\right) + \dots + \left(1 - \frac{2}{a_n}\right)$

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4. Let $a \ge b \ge c \ge d > 0$. Show that

$$\frac{b^3}{a} + \frac{c^3}{b} + \frac{d^3}{c} + \frac{a^3}{d} + 3\left(ab + bc + cd + da\right) \ge 4\left(a^2 + b^2 + c^2 + d^2\right).$$

Other problems (in Korean) are also available at https://www.facebook.com/KoreanMathOlympiad

– day 2

5 Given an integer $n \ge 2$, show that there exist two integers a, b which satisfy the following.

For all integer m, $m^3 + am + b$ is not a multiple of n.

6 Let triangle ABC be an acute scalene triangle, and denote D, E, F by the midpoints of BC, CA, AB, respectively. Let the circumcircle of DEF be O_1, and its center be N. Let the circumcircle of BCN be O_2. O_1 and O_2 meet at two points P, Q. O_2 meets AB at point $K(\neq B)$ and meets AC at point $L(\neq C)$. Show that the three lines EF, PQ, KL are concurrent.

7 Prove that there is no function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ satisfying:

$f(x + y^2) \ge f(x) + y$ for all two nonnegative real numbers x, y.

8 For a positive integer n, there is a school with n people. For a set X of students in this school, if any two students in X know each other, we call X *well-formed*. If the maximum number of students in a well-formed set is k, show that the maximum number of well-formed sets is not greater than $3^{(n+k)/3}$.

Here, an empty set and a set with one student is regarded as well-formed as well.