Peru IMO TST 2008

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– day 1

1 Let ABC be a triangle and let I be the incenter. Ia Ib and Ic are the excenters opposite to points A B and C respectively. Let La be the line joining the orthocenters of triangles IBC and $IaBC$. Define Lb and Lc in the same way.
Prove that La Lb and Lc are concurrent.

Daniel

2 Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(2f(x)+y)=f(f(x)-f(y))+2y+x,$$

for all $x,y\in\mathbb{R}$.

3 Given a positive integer n, consider the sequence (a_i), $1\leq i\leq 2n$, defined as follows:

$a_{2k-1}=-k, 1\leq k\leq n$ $a_{2k}=n-k+1, 1\leq k\leq n.$

We call a pair of numbers (b,c) good if the following conditions are met:

$i) 1\leq b<c\leq 2n,$

$ii) \sum_{j=b}^{c} a_j=0$

If $B(n)$ is the number of good pairs corresponding to n, prove that there are infinitely many n for which $B(n)=n$.

– day 2

4 Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two non-concentric circumferences such that $\mathcal{S}_1$ is inside $\mathcal{S}_2$. Let K be a variable point on $\mathcal{S}_1$. The line tangent to $\mathcal{S}_1$ at point K intersects $\mathcal{S}_2$ at points A and B. Let M be the midpoint of arc AB that is in the semiplane determined by AB that does not contain $\mathcal{S}_1$. Determine the locus of the point symmetric to M with respect to K.

5 When we cut a rope into two pieces, we say that the cut is special if both pieces have different lengths. We cut a chord of length 2008 into two pieces with integer lengths and we write those lengths on the board. Afterwards, we cut one of the pieces into two new pieces with integer lengths and we write those lengths on the board. This process ends until all pieces have length 1.

$a)$ Find the minimum possible number of special cuts. $b)$ Prove that, for all processes that have the minimum possible number of special cuts, the number of different integers on the board is always the same.

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6 We say that a positive integer is happy if can expressed in the form $(a^2b)/(a-b)$ where $a > b > 0$ are integers. We also say that a positive integer m is evil if it doesn't a happy integer n such that $d(n) = m$. Prove that all integers happy and evil are a power of 4.
