Peru IMO TST 2010

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by parmenides51, socrates, April

– day 1

1 Let ABC be an acute-angled triangle and F a point in its interior such that

$$\angle AFB = \angle BFC = \angle CFA = 120^\circ.$$

Prove that the Euler lines of the triangles AFB, BFC and CFA are concurrent.

2 A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.
(a) Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,. . ., $P(50)$ are balanced.
(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Proposed by Jorge Tipe, Peru

3 Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighbouring buckets, empties them to the river and puts them back. Then the next round begins. The Stepmother goal's is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Proposed by Gerhard Woeginger, Netherlands

– day 2

4 Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1+ab}{a+b} + \frac{1+bc}{b+c} + \frac{1+ca}{c+a} \geq 5.$$

5 Let $\mathbb{N}$ be the set of positive integers. For each subset $\mathcal{X}$ of $\mathbb{N}$ we define the set $\Delta(\mathcal{X})$ as the set of all numbers $|m - n|$, where m and n are elements of $\mathcal{X}$, ie:

$$\Delta(\mathcal{X}) = \{|m - n| \mid m, n \in \mathcal{X}\}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two infinite, disjoint sets whose union is $\mathbb{N}$.
a) Prove that the set $\Delta(\mathcal{A}) \cap \Delta(\mathcal{B})$ has infinitely many elements.
b) Prove that there exists an infinite subset $\mathcal{C}$ of $\mathbb{N}$ such that $\Delta(\mathcal{C})$ is a subset of $\Delta(\mathcal{A}) \cap \Delta(\mathcal{B})$.

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6 Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

Proposed by Eugene Bilopitov, Ukraine

– day 3

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7 Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} + 3(ab + bc + ca) \geq \frac{11}{2}.$$

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8 Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

Proposed by David Monk, United Kingdom

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9 Find all positive integers n such that there exists a sequence of positive integers a_1, a_2,. . ., a_n satisfying:

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

Proposed by North Korea
