Peru IMO TST 2011

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by parmenides51, socrates, Amir Hossein, orl

– day 1

1 Let $\mathbb{Z}^+$ denote the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ that satisfy the following condition: for each positive integer n, there exists a positive integer k such that

$$\sum_{i=1}^{k} f_i(n) = kn,$$

where $f_1(n) = f(n)$ and $f_{i+1}(n) = f(f_i(n))$, for $i \geq 1$.

2 Let $A_1A_2 \dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

Proposed by Nairi Sedrakyan, Armenia

3 Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n | P(m) - P(k)$ implies $n | m - k$ for all integers m, k. We say that (a, b) is $very\ good$ if (a, b) is n-good for infinitely many positive integers n.

-**(a)** Find a pair (a, b) which is 51-good, but not very good.

-**(b)** Show that all 2010-good pairs are very good.

Proposed by Okan Tekman, Turkey

– day 2

4 Let ABC be an acute triangle, and AA_1, BB_1, and CC_1 its altitudes. Let A_2 be a point on segment AA_1 such that $\angle BA_2C = 90^\circ$. The points B_2 and C_2 are defined similarly. Let A_3 be the intersection point of segments B_2C and BC_2. The points B_3 and C_3 are defined similarly. Prove that the segments A_2A_3, B_2B_3, and C_2C_3 are concurrent.

5 On some planet, there are 2^N countries $(N \geq 4)$. Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

Proposed by Toni Kokan, Croatia

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6 Let $a_1, a_2, \cdots, a_n$ be real numbers, with $n \geq 3$, such that $a_1 + a_2 + \cdots + a_n = 0$ and

$$2a_k \leq a_{k-1} + a_{k+1} \quad \text{for} \quad k = 2, 3, \cdots, n-1.$$

Find the least number $\lambda(n)$, such that for all $k \in \{1, 2, \cdots, n\}$ it is satisfied that $|a_k| \leq \lambda(n) \cdot \max\{|a_1|, |a_n|\}$.
