Peru IMO TST 2012

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–	day 1
1	Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $$f(f(x)) = \frac{x^2 - x}{2} \cdot f(x) + 2 - x,$$ for all $x \in \mathbb{R}$. Find all possible values of $f(2)$.
2	Let a, b, c be the lengths of the sides of a triangle, and h_a, h_b, h_c the lengths of the heights corresponding to the sides a, b, c, respectively. If $t \geq \frac{1}{2}$ is a real number, show that there is a triangle with sidelengths $$t \cdot a + h_a,\ t \cdot b + h_b,\ t \cdot c + h_c.$$
3	Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half. *Proposed by Gerhard Wginger, Austria*
–	day 2
4	An infinite triangular lattice is given, such that the distance between any two adjacent points is always equal to 1. Points A, B, and C are chosen on the lattice such that they are the vertices of an equilateral triangle of side length L, and the sides of ABC contain no points from the lattice. Prove that, inside triangle ABC, there are exactly $\frac{L^2-1}{2}$ points from the lattice.
5	Let $ABCD$ be a parallelogram such that $\angle ABC > 90^\circ$, and $\mathcal{L}$ the line perpendicular to BC that passes through B. Suppose that the segment CD does not intersect $\mathcal{L}$. Of all the circumferences that pass through C and D, there is one that is tangent to $\mathcal{L}$ at P, and there is another one that is tangent to $\mathcal{L}$ at Q (where $P \neq Q$). If M is the midpoint of AB, prove that $\angle PMD = \angle QMD$.
6	Let p be an odd prime number. For every integer a, define the number $S_a = \sum_{j=1}^{p-1} \frac{a^j}{j}$. Let $m, n \in \mathbb{Z}$, such that $S_3 + S_4 - 3S_2 = \frac{m}{n}$. Prove that p divides m. *Proposed by Romeo Metrovi, Montenegro*