Peru IMO TST 2014

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– pre - selection

1 a) Find at least two functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$2f(x^2)\geq xf(x)+x,$$

for all $x\in\mathbb{R}^+$.

b) Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a function such that

$$2f(x^2)\geq xf(x)+x,$$

for all $x\in\mathbb{R}^+$. Show that $f(x^3)\geq x^2$, for all $x\in\mathbb{R}^+$.

Can we find the best constant $a\in\mathbb{R}$ such that $f(x)\geq x^a$, for all $x\in\mathbb{R}^+$?

2 Let n be a positive integer. There is an infinite number of cards, each one of them having a non-negative integer written on it, such that for each integer $l\geq 0$, there are exactly n cards that have the number l written on them. A move consists of picking 100 cards from the infinite set of cards and discarding them. Find the least possible value of n for which there is an infinitely long series of moves such that for each positive integer k, the sum of the numbers written on the 100 chosen cards during the k-th move is equal to k.

3 Let ABC be an acuteangled triangle with $AB>BC$ inscribed in a circle. The perpendicular bisector of the side AC cuts arc AC, containing B, in Q. Let M be a point on the segment AB such that $AM=MB+BC$. Prove that the circumcircle of the triangle BMC cuts BQ in its midpoint.

4 A positive integer is called lonely if the sum of the reciprocals of its positive divisors (including 1 and itself) is different from the sum of the reciprocals of the positive divisors of any positive integer.
a) Prove that every prime number is lonely.
b) Prove that there are infinitely many positive integers that are not lonely.

– day 1

5 n vertices from a regular polygon with $2n$ sides are chosen and coloured red. The other n vertices are coloured blue. Afterwards, the $\binom{n}{2}$ lengths of the segments formed with all pairs of red vertices are ordered in a non-decreasing sequence, and the same procedure is done with the

$\binom{n}{2}$ lengths of the segments formed with all pairs of blue vertices. Prove that both sequences are identical.

6 Let ABC be a triangle where $AB > BC$, and D and E be points on sides AB and AC respectively, such that DE and AC are parallel. Consider the circumscribed circumference of triangle ABC. A circumference that passes through points D and E is tangent to the arc AC that does not contain B at point P. Let Q be the reflection of point P with respect to the perpendicular bisector of AC. The segments BQ and DE intersect at X. Prove that $AX = XC$.

7 Let n be a positive integer. Mariano divides a rectangle into n^2 smaller rectangles by drawing $n-1$ vertical lines and $n-1$ horizontal lines, parallel to the sides of the larger rectangle. On every step, Emilio picks one of the smaller rectangles and Mariano tells him its area. Find the least positive integer k for which it is possible that Emilio can do k conveniently thought steps in such a way that with the received information, he can determine the area of each one of the n^2 smaller rectangles.

– day 2

8 Let x, y, z be real numbers such that

$$\begin{cases} x^2 + y^2 + z^2 + (x+y+z)^2 = 9 \\ xyz \leq \frac{15}{32} \end{cases}$$

Find the maximum possible value of x.

9 Prove that for every positive integer n there exist integers a and b, both greater than 1, such that $a^2 + 1 = 2b^2$ and $a - b$ is a multiple of n.

10 Let $ABCDEF$ be a convex hexagon that does not have two parallel sides, such that $\angle AFB = \angle FDE, \angle DFE = \angle BDC$ and $\angle BFC = \angle ADF$. Prove that the lines AB, FC and DE are concurrent if and only if the lines AF, BE and CD are concurrent.

– day 3

11 Let ABC be a triangle, and P be a variable point inside ABC such that AP and CP intersect sides BC and AB at D and E respectively, and the area of the triangle APC is equal to the area of quadrilateral $BDPE$. Prove that the circumscribed circumference of triangle BDE passes through a fixed point different from B.

12 Every single point on the plane with integer coordinates is coloured either red, green or blue. Find the least possible positive integer n with the following property: no matter how the points are coloured, there is always a triangle with area n that has its 3 vertices with the same colour.

13 Let r be a positive integer and let N be the smallest positive integer such that the numbers $\frac{N}{n+r}\binom{2n}{n}$, $n = 0, 1, 2, \ldots$, are all integer. Show that $N = \frac{r}{2}\binom{2r}{r}$.

– day 4

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14 Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

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15 Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geq 1$. If

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_1 + n$$

and

$$a_{a_i} \leq n + i - 1 \quad \text{for} \quad i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \leq n^2.$$

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16 Let n be a positive integer, and let A be a subset of $\{1, \cdots, n\}$. An A-partition of n into k parts is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \cdots, a_k$ belong to A and are not necessarily distinct. The number of different parts in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \cdots, a_k\}$.
We say that an A-partition of n into k parts is optimal if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.
