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– pre - selection

1 Find all positive integers n for which there exist real numbers $x_1, x_2, ..., x_n$ satisfying all of the following conditions:
(i) $-1 < x_i < 1$, for all $1 \leq i \leq n$.
(ii) $x_1 + x_2 + ... + x_n = 0$.
(iii) $\sqrt{1-x_1^2} + \sqrt{1-x_2^2} + ... + \sqrt{1-x_n^2} = 1$.

2 Ana chose some unit squares of a 50×50 board and placed a chip on each of them. Prove that Beto can always choose at most 99 empty unit squares and place a chip on each so that each row and each column of the board contains an even number of chips.

3 Let M be the midpoint of the arc BAC of the circumcircle of the triangle ABC, I the incenter of the triangle ABC and L a point on the side BC such that AL is bisector. The line MI cuts the circumcircle again at K. The circumcircle of the triangle AKL cuts the line BC again at P. Prove that $\angle AIP = 90^\circ$.

4 Let $n \geq 2$ be an integer. The permutation $a_1, a_2, ..., a_n$ of the numbers $1, 2, ..., n$ is called *quadratic* if $a_i a_{i+1} + 1$ is a perfect square for all $1 \leq i \leq n-1$. The permutation $a_1, a_2, ..., a_n$ of the numbers $1, 2, ..., n$ is called *cubic* if $a_i a_{i+1} + 1$ is a perfect cube for all $1 \leq i \leq n-1$.
a) Prove that for infinitely many values of n is there at least one quadratic permutation of the numbers $1, 2, ..., n$.
b) Prove that for no value of n is there a cubic permutation of the numbers $1, 2, ..., n$.

– day 1

5 We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m .

Proposed by Abbas Mehrabian, Iran

6 Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

Proposed by Hong Kong

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7 For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \le i \le n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i-th step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \le cD$.

Proposed by Georgia

– day 2

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8 Let I be the incenter of the ABC triangle. The circumference that passes through I and has center
in A intersects the circumscribed circumference of the ABC triangle at points M and N. Prove that the line MN is tangent to the inscribed circle of the ABC triangle.

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9 Let A be a finite set of functions $f : \mathbb{R} \to \mathbb{R}$. It is known that: - If $f, g \in A$ then $f(g(x)) \in A$. - For all $f \in A$ there exists $g \in A$ such that $f(f(x) + y) = 2x + g(g(y) - x)$, for all $x, y \in \mathbb{R}$. Let $i : \mathbb{R} \to \mathbb{R}$ be the identity function, ie, $i(x) = x$ for all $x \in \mathbb{R}$. Prove that $i \in A$.

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10 A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.
Determine all possible first moves of the first player after which he has a winning strategy.

Proposed by Ilya Bogdanov & Vladimir Bragin, Russia

– day 3

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11 Let $n \geq 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\, 0 \leq k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n .

Proposed by Serbia

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12 Find the least positive real number α with the following property: if the weight of a finite number of pumpkins is 1 ton and the weight of each pumpkin is not greater than α tons then the pumpkins can be distributed in 50 boxes (some boxes can be empty) so that there is no more than α tons of pumpkins in each box.

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13 Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V , respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y . Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points $I, X,$ and Y are collinear, then the points $I, W,$ and Z are also collinear.

Proposed by David B. Rush, USA

– day 4

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14 Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{a_i^2}.$$

Prove that for every $i = 1, 2, \ldots, n$ we can find i numbers with sum at least i.

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15 Let ABC be a triangle. The points $K, L,$ and M lie on the segments $BC, CA,$ and AB, respectively, such that the lines $AK, BL,$ and CM intersect in a common point. Prove that it is possible to choose two of the triangles $ALM, BMK,$ and CKL whose inradii sum up to at least the inradius of the triangle ABC.

Proposed by Estonia

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16 Let $c \geq 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geq 1$. Prove that for each integer $n \geq 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

Proposed by Austria
