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Art of Problem Solving

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– pre - selection

1 **The real problem was:**

The positive real numbers a, b, c with $ab + bc + ca = 1$ Show that: $\sqrt{a + \frac{1}{a}} + \sqrt{b + \frac{1}{b}} + \sqrt{c + \frac{1}{c}}$
$2(\sqrt{a} + \sqrt{b} + \sqrt{c})$

2 Determine how many 100-positive integer sequences satisfy the two conditions following:
- At least one term of the sequence is equal to 4 or 5.
- Any two adjacent terms differ as a maximum in 2.

3 Let $ABCD$ a convex quadrilateral such that AD and BC are not parallel. Let M and N the midpoints of AD and BC respectively. The segment MN intersects AC and BD in K and L respectively, Show that at least one point of the intersections of the circumcircles of AKM and BNL is in the line AB.

4 Let N be the set of positive integers.
Find all the functions $f : N \to N$ with $f(1) = 2$ and such that $max\{f(m)+f(n), m+n\}$ divides $min\{2m + 2n, f(m + n) + 1\}$ for all m, n positive integers

– day 1

5 Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

6 Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f(x - f(y)) = f(f(x)) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

7 Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

– day 2

8 Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \ldots + a_n \geq n$ for every $n \geq 2$.

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9 Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called [i]k-good[/i] if $\gcd(f(m) + n, f(n) + m) \leq k$ for all $m \neq n$. Find all k such that there exists a k-good function.

Proposed by James Rickards, Canada

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10 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

– day 3

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11 Let $n > 2$ be an integer. A child has n^2 candies, which are distributed in n boxes. An operation consists in choosing two boxes that together contain an even number of candies and redistribute the candy from those boxes so that both contain the same amount of candy. Determine all the values of n for which the child, after some operations, can get each box containing n candies, no matter which the initial distribution of candies is.

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12 Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

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13 Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;
(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

Proposed by Ang Jie Jun, Singapore

– day 4

14 Determine all positive integers M such that the sequence $a_0, a_1, a_2, \cdots$ defined by

$$a_0 = M + \frac{1}{2} \qquad \text{and} \qquad a_{k+1} = a_k \lfloor a_k \rfloor \quad \text{for } k = 0, 1, 2, \cdots$$

contains at least one integer term.

15 Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \le n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.
(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.
(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

Proposed by Finland

16 Find all pairs (m, n) of positive integers that have the following property:
For every polynomial $P(x)$ of real coefficients and degree m, there exists a polynomial $Q(x)$ of real coefficients and degree n such that $Q(P(x))$ is divisible by $Q(x)$.