Peru IMO TST 2018

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by parmenides51, MarkBcc168, math90, fastlikearabbit, Jafet98

– pre - selection

1 A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Proposed by Jeck Lim, Singapore

2 Let $a_1, a_2, \dots a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \cdots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

3 Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

4 Find all pairs (p, q) of prime numbers which $p > q$ and

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

– day 1

5 Let d be a positive integer.
The seqeunce $a_1, a_2, a_3, \dots$ of positive integers is defined by $a_1 = 1$ and $a_{n+1} = n \left\lfloor \frac{a_n}{n} \right\rfloor + d$ for $n = 1, 2, 3, \dots$.
Prove that there exists a positive integer N so that the terms $a_N, a_{N+1}, a_{N+2}, \dots$ form an arithmetic progression.

Note: If x is a real number, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

6 Let n be a positive integer. Define a chameleon to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters $a, b,$ and c. Define a swap to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X , there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

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7 Let ABC be, with $AC > AB$, an acute-angled triangle with circumcircle Γ and M the midpoint of side BC. Let N be a point in the interior of $\triangle ABC$. Let D and E be the feet of the perpendiculars from N to AB and AC, respectively. Suppose that $DE \perp AM$. The circumcircle of $\triangle ADE$ meets Γ at L ($L \neq A$), lines AL and DE intersects at K and line AN meets Γ at F ($F \neq A$). Prove that if N is the midpoint of the segment AF then $KA = KF$.

– day 2

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8 You want to paint some edges of a regular dodecahedron red so that each face has an even number of painted edges (which can be zero). Determine from How many ways this coloration can be done.

Note: A regular dodecahedron has twelve pentagonal faces and in each vertex concur three edges. The edges of the dodecahedron are all different for the purpose of the coloring . In this way, two colorings are the same only if the painted edges they are the same.

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9 A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all} \quad n > 2017.$$

Prove that the sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.

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10 For each positive integer $m > 1$, let $P(m)$ be the product of all prime numbers that divide m. Define the sequence $a_1, a_2, a_3, ...$ as followed: $a_1 > 1$ is an arbitrary positive integer, $a_{n+1} = a_n + P(a_n)$ for each positive integer n.
Prove that there exist positive integers j and k such that a_j is the product of the first k prime numbers.
