Federal Competition For Advanced Students, Part 1, 2018

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by sqing, parmenides51

1 Let α be an arbitrary positive real number. Determine for this number α the greatest real number C such that the inequality

$$\left(1+\frac{\alpha}{x^2}\right)\left(1+\frac{\alpha}{y^2}\right)\left(1+\frac{\alpha}{z^2}\right)\geq C\left(\frac{x}{z}+\frac{z}{x}+2\right)$$

is valid for all positive real numbers x, y and z satisfying $xy+yz+zx=\alpha$. When does equality occur?

(Proposed by Walther Janous)

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2 Let ABC be a triangle with incenter I. The incircle of the triangle is tangent to the sides BC and AC in points D and E, respectively. Let P denote the common point of lines AI and DE, and let M and N denote the midpoints of sides BC and AB, respectively. Prove that points M, N and P are collinear.

(Proposed by Karl Czakler)

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3 Alice and Bob determine a number with 2018 digits in the decimal system by choosing digits from left to right. Alice starts and then they each choose a digit in turn. They have to observe the rule that each digit must differ from the previously chosen digit modulo 3. Since Bob will make the last move, he bets that he can make sure that the final number is divisible by 3. Can Alice avoid that?

(Proposed by Richard Henner)

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4 Let M be a set containing positive integers with the following three properties:
(1) $2018 \in M$.
(2) If $m \in M$, then all positive divisors of m are also elements of M.
(3) For all elements $k, m \in M$ with $1 < k < m$, the number $km+1$ is also an element of M.
Prove that $M = Z_{\geq 1}$.

(Proposed by Walther Janous)
