Czech And Slovak Mathematical Olympiad, Round III, Category A 2019

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by byk7

1 Find all triplets $(x, y, z) \in \mathbb{R}^3$ such that

$$x^2 - yz = |y - z| + 1,$$
$$y^2 - zx = |z - x| + 1,$$
$$z^2 - xy = |x - y| + 1.$$

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2 Let be $ABCD$ a rectangle with $|AB| = a \geq b = |BC|$. Find points P, Q on the line BD such that $|AP| = |PQ| = |QC|$. Discuss the solvability with respect to the lengths a, b.

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3 Let a, b, c, n be positive integers such that the following conditions hold
(i) numbers $a, b, c, a + b + c$ are pairwise coprime,
(ii) number $(a + b)(b + c)(c + a)(a + b + c)(ab + bc + ca)$ is a perfect n-th power.
Prove, that the product abc can be expressed as a difference of two perfect n-th powers.

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4 Let be ABC an acute-angled triangle. Consider point P lying on the opposite ray to the ray BC such that $|AB| = |BP|$. Similarly, consider point Q on the opposite ray to the ray CB such that $|AC| = |CQ|$. Denote J the excenter of ABC with respect to A and D, E tangent points of this excircle with the lines AB and AC, respectively. Suppose that the opposite rays to DP and EQ intersect in $F \neq J$. Prove that $AF \perp FJ$.

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5 Prove that there are infinitely many integers which cannot be expressed as $2^a + 3^b - 5^c$ for non-negative integers a, b, c.

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6 Assume we can fill a table $n \times n$ with all numbers $1, 2, \ldots, n^2 - 1, n^2$ in such way that arithmetic means of numbers in every row and every column is an integer. Determine all such positive integers n.
