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by ComplexPhi

–	**Day 1**
1	Let ABC be a triangle, let O be its circumcenter, let A' be the orthogonal projection of A on the line BC, and let X be a point on the open ray AA' emanating from A. The internal bisectrix of the angle BAC meets the circumcircle of ABC again at D. Let M be the midpoint of the segment DX. The line through O and parallel to the line AD meets the line DX at N. Prove that the angles BAM and CAN are equal.
2	Let ABC be a triangle, and let r denote its inradius. Let R_A denote the radius of the circle internally tangent at A to the circle ABC and tangent to the line BC; the radii R_B and R_C are defined similarly. Show that $\frac{1}{R_A}+\frac{1}{R_B}+\frac{1}{R_C} \leq \frac{2}{r}$.
3	A Pythagorean triple is a solution of the equation $x^2+y^2=z^2$ in positive integers such that $x<y$. Given any non-negative integer n , show that some positive integer appears in precisely n distinct Pythagorean triples.
4	Let k be a positive integer congruent to 1 modulo 4 which is not a perfect square and let $a=\frac{1+\sqrt{k}}{2}$. Show that $\{\left\lfloor a^2n\right\rfloor-\lfloor a\left\lfloor an\right\rfloor\rfloor : n \in \mathbb{N}_{>0}\}=\{1,2,\ldots,\lfloor a\rfloor\}$.
5	Given an integer $N \geq 4$, determine the largest value the sum $$\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor+1}\left(\left\lfloor\frac{n_i}{2}\right\rfloor+1\right)$$ may achieve, where $k, n_1, \ldots, n_k$ run through the integers subject to $k \geq 3$, $n_1 \geq \ldots \geq n_k \geq 1$ and $n_1+\ldots+n_k=N$.
–	**Day 2**
1	Let a be an integer and n a positive integer . Show that the sum : $$\sum_{k=1}^{n} a^{(k,n)}$$ is divisible by n , where (x,y) is the greatest common divisor of the numbers x and y .

2 Let ABC be a triangle . Let A' be the center of the circle through the midpoint of the side BC and the orthogonal projections of B and C on the lines of support of the internal bisectrices of the angles ACB and ABC , respectively ; the points B' and C' are defined similarly . Prove that the nine-point circle of the triangle ABC and the circumcircle of $A'B'C'$ are concentric.

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3 Given a positive real number t , determine the sets A of real numbers containing t , for which there exists a set B of real numbers depending on A , $|B| \geq 4$, such that the elements of the set $AB = \{ab \mid a \in A, b \in B\}$ form a finite arithmetic progression .

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4 Consider the integral lattice $\mathbb{Z}^n$, $n \geq 2$, in the Euclidean n-space. Define a *line* in $\mathbb{Z}^n$ to be a set of the form $a_1 \times \cdots \times a_{k-1} \times \mathbb{Z} \times a_{k+1} \times \cdots \times a_n$ where k is an integer in the range $1, 2, \ldots, n$, and the a_i are arbitrary integers. A subset A of $\mathbb{Z}^n$ is called *admissible* if it is non-empty, finite, and every *line* in $\mathbb{Z}^n$ which intersects A contains at least two points from A. A subset N of $\mathbb{Z}^n$ is called *null* if it is non-empty, and every *line* in $\mathbb{Z}^n$ intersects N in an even number of points (possibly zero).
(a) Prove that every *admissible* set in $\mathbb{Z}^2$ contains a *null* set.
(b) Exhibit an *admissible* set in $\mathbb{Z}^3$ no subset of which is a *null* set .

– **Day 3**

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1 Two circles γ and γ' cross one another at points A and B . The tangent to γ' at A meets γ again at C , the tangent to γ at A meets γ' again at C' , and the line CC' separates the points A and B . Let Γ be the circle externally tangent to γ , externally tangent to γ' , tangent to the line CC', and lying on the same side of CC' as B . Show that the circles γ and γ' intercept equal segments on one of the tangents to Γ through A .

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2 Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences of real numbers such that $a_0 > \frac{1}{2}$, $a_{n+1} \geq a_n$ and $b_{n+1} = a_n(b_n + b_{n+2})$ for all non-negative integers n . Show that the sequence $(b_n)_{n\geq 0}$ is bounded .

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3 If k and n are positive integers , and $k \leq n$, let $M(n,k)$ denote the least common multiple of the numbers $n, n-1, \ldots, n-k+1$.Let $f(n)$ be the largest positive integer $k \leq n$ such that $M(n,1) < M(n,2) < \ldots < M(n,k)$. Prove that :
(a) $f(n) < 3\sqrt{n}$ for all positive integers n .
(b) If N is a positive integer , then $f(n) > N$ for all but finitely many positive integers n.

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4 Given two integers $h \geq 1$ and $p \geq 2$, determine the minimum number of pairs of opponents an hp-member parliament may have, if in every partition of the parliament into h houses of p member each, some house contains at least one pair of opponents.

– **Day 4**

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1 Let ABC and ABD be coplanar triangles with equal perimeters. The lines of support of the internal bisectrices of the angles CAD and CBD meet at P. Show that the angles APC and

BPD are congruent.

2 Given an integer $k \geq 2$, determine the largest number of divisors the binomial coefficient $\binom{n}{k}$ may have in the range $n-k+1, \ldots, n$, as n runs through the integers greater than or equal to k.

3 Let n be a positive integer . If σ is a permutation of the first n positive integers , let $S(\sigma)$ be the set of all distinct sums of the form $\sum_{i=k}^{l} \sigma(i)$ where $1 \leq k \leq l \leq n$.

(a) Exhibit a permutation σ of the first n positive integers such that $|S(\sigma)| \geq \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$.

(b) Show that $|S(\sigma)| > \frac{n\sqrt{n}}{4\sqrt{2}}$ for all permutations σ of the first n positive integers .

– **Day 5**

1 Let ABC be a triangle. Let P_1 and P_2 be points on the side AB such that P_2 lies on the segment BP_1 and $AP_1 = BP_2$; similarly, let Q_1 and Q_2 be points on the side BC such that Q_2 lies on the segment BQ_1 and $BQ_1 = CQ_2$. The segments P_1Q_2 and P_2Q_1 meet at R, and the circles P_1P_2R and Q_1Q_2R meet again at S, situated inside triangle P_1Q_1R. Finally, let M be the midpoint of the side AC. Prove that the angles P_1RS and Q_1RM are equal.

2 Let n be an integer greater than 1, and let p be a prime divisor of n. A confederation consists of p states, each of which has exactly n airports. There are p air companies operating interstate flights only such that every two airports in different states are joined by a direct (two-way) flight operated by one of these companies. Determine the maximal integer N satisfying the following condition: In every such confederation it is possible to choose one of the p air companies and N of the np airports such that one may travel (not necessarily directly) from any one of the N chosen airports to any other such only by flights operated by the chosen air company.

3 Define a sequence of integers by $a_0 = 1$, and $a_n = \sum_{k=0}^{n-1} \binom{n}{k} a_k$, $n \geq 1$. Let m be a positive integer , let p be a prime , and let q and r be non-negative integers . Prove that :

$$a_{p^m q + r} \equiv a_{p^{m-1} q + r} \pmod{p^m}$$