German National Olympiad 2019, Final Round

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–	Day 1
1	Determine all real solutions (x,y) of the following system of equations: $$x = 3x^2y - y^3, \\ y = x^3 - 3xy^2$$
2	Let a and b be two circles, intersecting in two distinct points Y and Z. A circle k touches the circles a and b externally in the points A and B. Show that the angular bisectors of the angles $\angle ZAY$ and $\angle YBZ$ intersect on the line YZ.
3	In the cartesian plane consider rectangles with sides parallel to the coordinate axes. We say that one rectangle is *below* another rectangle if there is a line g parallel to the x-axis such that the first rectangle is below g, the second one above g and both rectangles do not touch g. Similarly, we say that one rectangle is *to the right of* another rectangle if there is a line h parallel to the y-axis such that the first rectangle is to the right of h, the second one to the left of h and both rectangles do not touch h. Show that any finite set of n pairwise disjoint rectangles with sides parallel to the coordinate axes can be enumerated as a sequence $(R_1,\dots,R_n)$ so that for all indices i,j with $1 \le i < j \le n$ the rectangle R_i is to the right of or below the rectangle R_j
–	Day 2
4	Show that for each non-negative integer n there are unique non-negative integers x and y such that we have $$n = \frac{(x+y)^2 + 3x + y}{2}.$$
5	We are given two positive integers p and q. Step by step, a rope of length 1 is cut into smaller pieces as follows: In each step all the currently longest pieces are cut into two pieces with the ratio $p:q$ at the same time. After an unknown number of such operations, the currently longest pieces have the length x.

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Determine in terms of x the number $a(x)$ of different lengths of pieces of rope existing at that time.

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6 Suppose that real numbers x, y and z satisfy the following equations:

$$x + \frac{y}{z} = 2,$$
$$y + \frac{z}{x} = 2,$$
$$z + \frac{x}{y} = 2.$$

Show that $s = x + y + z$ must be equal to 3 or 7.

Note: It is not required to show the existence of such numbers x, y, z.

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