ELMO Problems 2019

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– Day 1

1 Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 1$, and let $c > 1$ be an integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \ge 0$. Show that there are infinitely many positive integers n such that $\gcd(x_n, n + c) = 1$.

Proposed by Milan Haiman and Carl Schildkraut

2 Let $m, n \ge 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid.* He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m - 1$ points that divide segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m, n) can Carl necessarily accomplish his task, regardless of which n points he is given?

*Here, the *centroid* of n points with coordinates $(x_1, y_1), \dots, (x_n, y_n)$ is the point with coordinates $\left(\frac{x_1+\cdots+x_n}{n}, \frac{y_1+\cdots+y_n}{n}\right)$.

Proposed by Holden Mui and Carl Schildkraut

3 Let $n \ge 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \dots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.

Proposed by Carl Schildkraut and Colin Tang

– Day 2

4 Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Proposed by Vincent Huang

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5 Let S be a nonempty set of positive integers such that, for any (not necessarily distinct) integers a and b in S, the number $ab+1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Proposed by Carl Schildkraut

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6 Carl chooses a *functional expression** E which is a finite nonempty string formed from a set $x_1, x_2, \dots$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1, x_2, \dots$. (For example, if Carl chooses the functional equation

$$f(2f(x_1)+x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.

(a) Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Carl can choose his functional equation such that S is nonempty but $S \subseteq X$.

(b) Can Carl choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

*These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any positive integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V+W$, $V-W$, and $V \cdot W$.

Proposed by Carl Schildkraut
