ELMO Shortlist 2019

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by pieater314159, tastymath75025

– Algebra

A1 Let a, b, c be positive reals such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$. Show that

$$a^abc+b^bca+c^cab\ge 27bc+27ca+27ab.$$

Proposed by Milan Haiman

A2 Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for all surjective functions $g:\mathbb{Z}\to\mathbb{Z}$, $f+g$ is also surjective. (A function g is surjective over $\mathbb{Z}$ if for all integers y, there exists an integer x such that $g(x)=y$.)

Proposed by Sean Li

A3 Let $m,n\ge 2$ be integers. Carl is given n marked points in the plane and wishes to mark their centroid.* He has no standard compass or straightedge. Instead, he has a device which, given marked points A and B, marks the $m-1$ points that divide segment $\overline{AB}$ into m congruent parts (but does not draw the segment).

For which pairs (m,n) can Carl necessarily accomplish his task, regardless of which n points he is given?

*Here, the *centroid* of n points with coordinates $(x_1,y_1),\dots,(x_n,y_n)$ is the point with coordinates $\left(\frac{x_1+\dots+x_n}{n},\frac{y_1+\dots+y_n}{n}\right)$.

Proposed by Holden Mui and Carl Schildkraut

A4 Find all nondecreasing functions $f:\mathbb{R}\to\mathbb{R}$ such that, for all $x,y\in\mathbb{R}$,

$$f(f(x))+f(y)=f(x+f(y))+1.$$

Proposed by Carl Schildkraut

A5 Carl chooses a *functional expression** E which is a finite nonempty string formed from a set $x_1,x_2,\dots$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E=0$, and lets S denote the set of functions $f:\mathbb{R}\to\mathbb{R}$ such that the equation holds for any choices of real numbers $x_1,x_2,\dots$. (For example, if Carl chooses the functional equation

$$f(2f(x_1) + x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.

(a) Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Carl can choose his functional equation such that S is nonempty but $S \subseteq X$.

(b) Can Carl choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

*These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any positive integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V + W$, $V - W$, and $V \cdot W$.

Proposed by Carl Schildkraut

– Combinatorics

C1 Elmo and Elmo's clone are playing a game. Initially, $n \geq 3$ points are given on a circle. On a player's turn, that player must draw a triangle using three unused points as vertices, without creating any crossing edges. The first player who cannot move loses. If Elmo's clone goes first and players alternate turns, who wins? (Your answer may be in terms of n.)

Proposed by Milan Haiman

C2 Adithya and Bill are playing a game on a connected graph with $n > 2$ vertices, two of which are labeled A and B, so that A and B are distinct and non-adjacent and known to both players. Adithya starts on vertex A and Bill starts on B. Each turn, both players move simultaneously: Bill moves to an adjacent vertex, while Adithya may either move to an adjacent vertex or stay at his current vertex. Adithya loses if he is on the same vertex as Bill, and wins if he reaches B alone. Adithya cannot see where Bill is, but Bill can see where Adithya is. Given that Adithya has a winning strategy, what is the maximum possible number of edges the graph may have? (Your answer may be in terms of n.)

Proposed by Steven Liu

C3 In the game of *Ring Mafia*, there are 2019 counters arranged in a circle. 673 of these counters are mafia, and the remaining 1346 counters are town. Two players, Tony and Madeline, take turns with Tony going first. Tony does not know which counters are mafia but Madeline does.

On Tonys turn, he selects any subset of the counters (possibly the empty set) and removes all counters in that set. On Madelines turn, she selects a town counter which is adjacent to a mafia counter and removes it. Whenever counters are removed, the remaining counters are brought closer together without changing their order so that they still form a circle. The game ends when either all mafia counters have been removed, or all town counters have been removed.

Is there a strategy for Tony that guarantees, no matter where the mafia counters are placed and what Madeline does, that at least one town counter remains at the end of the game?

Proposed by Andrew Gu

C4 Let $n \geq 3$ be a fixed integer. A game is played by n players sitting in a circle. Initially, each player draws three cards from a shuffled deck of $3n$ cards numbered $1, 2, \ldots, 3n$. Then, on each turn, every player simultaneously passes the smallest-numbered card in their hand one place clockwise and the largest-numbered card in their hand one place counterclockwise, while keeping the middle card.

Let T_r denote the configuration after r turns (so T_0 is the initial configuration). Show that T_r is eventually periodic with period n, and find the smallest integer m for which, regardless of the initial configuration, $T_m = T_{m+n}$.

Proposed by Carl Schildkraut and Colin Tang

C5 Given a permutation of $1, 2, 3, \ldots, n$, with consecutive elements a, b, c (in that order), we may perform either of the *moves*:

- If a is the median of a, b, and c, we may replace a, b, c with b, c, a (in that order)
- If c is the median of a, b, and c, we may replace a, b, c with c, a, b (in that order)

What is the least number of sets in a partition of all $n!$ permutations, such that any two permutations in the same set are obtainable from each other by a sequence of moves?

Proposed by Milan Haiman

– Geometry

G1 Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Proposed by Luke Robitaille

G2 Carl is given three distinct non-parallel lines ℓ_1, ℓ_2, ℓ_3 and a circle ω in the plane. In addition to a normal straightedge, Carl has a special straightedge which, given a line ℓ and a point P, constructs a new line passing through P parallel to ℓ. (Carl does not have a compass.) Show that Carl can construct a triangle with circumcircle ω whose sides are parallel to ℓ_1, ℓ_2, ℓ_3 in some order.

Proposed by Vincent Huang

G3 Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides $BC, CA,$ and AB at $D, E,$ and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Proposed by Ankit Bisain

G4 Let triangle ABC have altitudes BE and CF which meet at H. The reflection of A over BC is A'. Let (ABC) meet $(AA'E)$ at P and $(AA'F)$ at Q. Let BC meet PQ at R. Prove that $EF \parallel HR$.

Proposed by Daniel Hu

G5 Given a triangle ABC for which $\angle BAC \neq 90^\circ$, let B_1, C_1 be variable points on AB, AC, respectively. Let B_2, C_2 be the points on line BC such that a spiral similarity centered at A maps B_1C_1 to C_2B_2. Denote the circumcircle of AB_1C_1 by ω. Show that if B_1B_2 and C_1C_2 concur on ω at a point distinct from B_1 and C_1, then ω passes through a fixed point other than A.

Proposed by Max Jiang

G6 Let ABC be an acute scalene triangle and let P be a point in the plane. For any point $Q \neq A, B, C$, define T_A to be the unique point such that $\triangle T_ABP \sim \triangle T_AQC$ and $\triangle T_ABP, \triangle T_AQC$ are oriented in the same direction (clockwise or counterclockwise). Similarly define T_B, T_C.

a) Find all P such that there exists a point Q with T_A, T_B, T_C all lying on the circumcircle of $\triangle ABC$. Call such a pair (P, Q) a *tasty pair* with respect to $\triangle ABC$.

b) Keeping the notations from a), determine if there exists a tasty pair which is also tasty with respect to $\triangle T_AT_BT_C$.

Proposed by Vincent Huang

– Number Theory

N1 Let $P(x)$ be a polynomial with integer coefficients such that $P(0) = 1$, and let $c > 1$ be an integer. Define $x_0 = 0$ and $x_{i+1} = P(x_i)$ for all integers $i \geq 0$. Show that there are infinitely many positive integers n such that $\gcd(x_n, n + c) = 1$.

Proposed by Milan Haiman and Carl Schildkraut

N2 Let $f : \mathbb{N} \to \mathbb{N}$. Show that $f(m) + n \mid f(n) + m$ for all positive integers $m \leq n$ if and only if $f(m) + n \mid f(n) + m$ for all positive integers $m \geq n$.

Proposed by Carl Schildkraut

N3 Let S be a nonempty set of positive integers such that, for any (not necessarily distinct) integers a and b in S, the number $ab + 1$ is also in S. Show that the set of primes that do not divide any element of S is finite.

Proposed by Carl Schildkraut

N4 A positive integer b and a sequence $a_0, a_1, a_2, \dots$ of integers $0 \le a_i < b$ is given. It is known that $a_0 \neq 0$ and the sequence $\{a_i\}$ is eventually periodic but has infinitely many nonzero terms. Let S be the set of positive integers n so that $n \mid (a_0 a_1 \dots a_n)_b$. Given that S is infinite, show that there are infinitely many primes that divide at least one element of S.

Proposed by Carl Schildkraut and Holden Mui

N5 Given an even positive integer m, find all positive integers n for which there exists a bijection $f : [n] \to [n]$ so that, for all $x, y \in [n]$ for which $n \mid mx - y$,

$$(n+1) \mid f(x)^m - f(y).$$

Note: For a positive integer n, we let $[n] = \{1, 2, \dots, n\}$.

Proposed by Milan Haiman and Carl Schildkraut