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220 There are 50 exact watches lying on a table.
Prove that there exist a certain moment, when the sum of the distances from the centre of the table to the ends of the minute hands is more than the sum of the distances from the centre of the table to the centres of the watches.

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221 A row of 1000 numbers is written on the blackboard. We write a new row, below the first according to the rule:
We write under every number a the natural number, indicating how many times the number a is encountered in the first line. Then we write down the third line: under every number b – the natural number, indicating how many times the number b is encountered in the second line, and so on.
a) Prove that there is a line that coincides with the preceding one.
b) Prove that the eleventh line coincides with the twelfth.
c) Give an example of the initial line such, that the tenth row differs from the eleventh.

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222 Given three circumferences of the same radius in a plane.
a) All three are crossing in one point K. Consider three arcs AK, CK, EK : the A, C, E are the points of the circumferences intersection and the arcs are taken in the clockwise direction. Every arc is inside one circle, outside the second and on the border of the third one. Prove that the sum of the arcs is 180 degrees.
b) Consider the case, when the three circles give a curvilinear triangle BDF as their intersection (instead of one point K). The arcs are taken in the clockwise direction. Every arc is inside one circle, outside the second and on the border of the third one. Prove that the sum of the AB, CD and EF arcs is 180 degrees.

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223 The natural numbers x_1 and x_2 are less than 1000. We build a sequence: $x_3 = |x_1 - x_2|$, $x_4 = min\{|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|\}$, ... $x_k = min\{|x_i - x_j|, 0 < i < j < k\}$, ...
Prove that $x_{21} = 0$.

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224 Can you mark the cube's vertices with the three-digit binary numbers in such a way, that the numbers at all the possible couples of neighbouring vertices differ in at least two digits?

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225 Given 4 vectors a, b, c, d in the plane, such that $a + b + c + d = 0$.
Prove the following inequality: $|a| + |b| + |c| + |d| \geq |a + d| + |b + d| + |c + d|$

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226 Given right 1976-angle. The middles of all the sides and diagonals are marked.
What is the greatest number of the marked points lying on one circumference?

227 There are n rectangles drawn on the rectangular sheet of paper with the sides of the rectangles parallel to the sheet sides. The rectangles do not have pairwise common interior points. Prove that after cutting out the rectangles the sheet will split into not more than $n+1$ part.

228 There are three straight roads. Three pedestrians are moving along those roads, and they are NOT on one line in the initial moment. Prove that they will be one line not more than twice

229 Given a chess-board 99×99 with a set F of fields marked on it (the set is different in three tasks). There is a beetle sitting on every field of the set F. Suddenly all the beetles have raised into the air and flied to another fields of the same set. The beetles from the neighbouring fields have landed either on the same field or on the neighbouring ones (may be far from their starting point). (We consider the fields to be neighbouring if they have at least one common vertex.)
Consider a statement:
"There is a beetle, that either stayed on the same field or moved to the neighbouring one".
Is it always valid if the figure F is:
a) A central cross, i.e. the union of the 50-th row and the 50-th column?
b) A window frame, i.e. the union of the 1-st, 50-th and 99-th rows and the 1-st, 50-th and 99-th columns?
c) All the chess-board?

230 Let us call "*big*" a triangle with all sides longer than 1. Given a equilateral triangle with all the sides equal to 5.
Prove that:
a) You can cut 100 *big* triangles out of given one.
b) You can divide the given triangle onto 100 *big* nonintersecting ones fully covering the initial one.
c) The same as b), but the triangles either do not have common points, or have one common side, or one common vertex.
d) The same as c), but the initial triangle has the side 3.

231 Given natural n. We shall call "universal" such a sequence of natural number $a_1, a_2, ..., a_k, k \geq n$, if we can obtain every transposition of the first n natural numbers (i.e such a sequence of n numbers, that every one is encountered only once) by deleting some its members. (Examples: $(1,2,3,1,2,1,3)$ is universal for $n = 3$, and $(1,2,3,2,1,3,1)$ – not, because you can't obtain $(3,1,2)$ from it.) The goal is to estimate the length of the shortest universal sequence for given n.
a) Give an example of the universal sequence of $n2$ members.
b) Give an example of the universal sequence of $(n^2 - n + 1)$ members.
c) Prove that every universal sequence contains not less than $n(n+1)/2$ members
d) Prove that the shortest universal sequence for $n = 4$ contains 12 members
e) Find as short universal sequence, as you can. The Organising Committee knows the method

for $(n^2 - 2n + 4)$ members.

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232 n numbers are written down along the circumference. Their sum equals to zero, and one of them equals 1.
a) Prove that there are two neighbours with their difference not less than $n/4$.
b) Prove that there is a number that differs from the arithmetic mean of its two neighbours not less than on $8/(n^2)$.
c) Try to improve the previous estimation, i.e what number can be used instead of 8?
d) Prove that for $n = 30$ there is a number that differs from the arithmetic mean of its two neighbours not less than on $2/113$, give an example of such 30 numbers along the circumference, that not a single number differs from the arithmetic mean of its two neighbours more than on $2/113$.

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233 Given right n-angle wit the point O – its centre. All the vertices are marked either with $+1$ or -1. We may change all the signs in the vertices of right k-angle ($2 \le k \le n$) with the same centre O. (By 2-angle we understand a segment, being halved by O.)
Prove that in a), b) and c) cases there exists such a set of $(+1)$s and (-1)s, that we can never obtain a set of $(+1)$s only.
a) $n = 15$,
b) $n = 30$,
c) $n > 2$,
d) Let us denote $K(n)$ the maximal number of $(+1)$ and (-1) sets such, that it is impossible to obtain one set from another. Prove, for example, that $K(200) = 2^{80}$

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234 Given a sphere of unit radius with the big circle (i.e of unit radius) that will be called ”equator”. We shall use the words ”pole”, ”parallel”,”meridian” as self-explanatory.
a) Let $g(x)$, where x is a point on the sphere, be the distance from this point to the equator plane.
Prove that $g(x)$ has the property if x_1, x_2, x_3 are the ends of the pairwise orthogonal radiuses, then $g(x_1)^2 + g(x_2)2 + g(x_3)^2 = 1$. (*)

Let function $f(x)$ be an arbitrary nonnegative function on a sphere that satisfies (*) property.
b) Let x_1 and x_2 points be on the same meridian between the north pole and equator, and x_1 is closer to the pole than x_2. Prove that $f(x_1) > f(x_2)$.
c) Let y_1 be closer to the pole than y_2. Prove that $f(y_1) > f(y_2)$.
d) Let z_1 and z_2 be on the same parallel. Prove that $f(z_1) = f(z_2)$.
e) Prove that for all $x, f(x) = g(x)$.
