All Soviet Union Mathematical Olympiad 1977

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235 Given a closed broken line without self-intersections in a plane. Not a triple of its vertices belongs to one straight line. Let us call ”special” a couple of line’s segments if the one’s extension intersects another. Prove that there is even number of special pairs.

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236 Given several points, not all lying on one straight line. Some number is assigned to every point. It is known, that if a straight line contains two or more points, than the sum of the assigned to those points equals zero. Prove that all the numbers equal to zero.

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237 a) Given a circle with two inscribed triangles T_1 and T_2. The vertices of T_1 are the middles of the arcs with the ends in the vertices of T_2. Consider a hexagon – the intersection of T_1 and T_2. Prove that its main diagonals are parallel to T_1 sides and are intersecting in one point.
b) The segment, that connects the middles of the arcs AB and AC of the circle circumscribed around the ABC triangle, intersects $[AB]$ and $[AC]$ sides in D and K points. Prove that the points A, D, K and O – the centre of the circle – are the vertices of a diamond.

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238 Several black and white checkers (tokens?) are standing along the circumference. Two men remove checkers in turn. The first removes all the black ones that had at least one white neighbour, and the second – all the white ones that had at least one black neighbour. They stop when all the checkers are of the same colour.
a) Let there be 40 checkers initially. Is it possible that after two moves of each man there will remain only one (checker)?
b) Let there be 1000 checkers initially. What is the minimal possible number of moves to reach the position when there will remain only one (checker)?

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239 Given infinite sequence a_n. It is known that the limit of $b_n = a_{n+1} - a_n/2$ equals zero.
Prove that the limit of a_n equals zero.

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240 There are direct routes from every city of a certain country to every other city. The prices are known in advance. Two tourists (they do not necessary start from one city) have decided to visit all the cities, using only direct travel lines. The first always chooses the cheapest ticket to the city, he has never been before (if there are several – he chooses arbitrary destination among the cheapests). The second – the most expensive (they do not return to the first city).
Prove that the first will spend not more money for the tickets, than the second.

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241 Every vertex of a convex polyhedron belongs to three edges. It is possible to circumscribe a circle around all its faces.
Prove that the polyhedron can be inscribed in a sphere.

242 The polynomial $x^{10}+?x^9+?x^8+...+?x+1$ is written on the blackboard. Two players substitute (real) numbers instead of one of the question marks in turn. (9 turns total.) The first wins if the polynomial will have no real roots.
Who wins?

243 Seven elves are sitting at a round table. Each elf has a cup. Some cups are filled with some milk. Each elf in turn and clockwise divides all his milk between six other cups. After the seventh has done this, every cup was containing the initial amount of milk. How much milk did every cup contain, if there was three litres of milk total?

244 Let us call "fine" the $2n$-digit number if it is exact square itself and the two numbers represented by its first n digits (first digit may not be zero) and last n digits (first digit may be zero, but it may not be zero itself) are exact squares also.
a) Find all two- and four-digit fine numbers.
b) Is there any six-digit fine number?
c) Prove that there exists 20-digit fine number.
d) Prove that there exist at least ten 100-digit fine numbers.
e) Prove that there exists 30-digit fine number.

245 Given a set of n positive numbers. For each its nonempty subset consider the sum of all the subset's numbers.
Prove that you can divide those sums onto n groups in such a way, that the least sum in every group is not less than a half of the greatest sum in the same group.

246 There are 1000 tickets with the numbers $000, 001, ..., 999$, and 100 boxes with the numbers $00, 01, ..., 99$. You may put a ticket in a box, if you can obtain the box number from the ticket number by deleting one digit. Prove that:
a) You can put all the tickets in 50 boxes;
b) 40 boxes is not enough for that;
c) it is impossible to use less than 50 boxes.
d) Consider 10000 4-digit tickets, and you are allowed to delete two digits. Prove that 34 boxes is enough for storing all the tickets.
e) What is the minimal used boxes set in the case of k-digit tickets?

247 Given a square 100×100 on the sheet of cross-lined paper. There are several broken lines drawn inside the square. Their links consist of the small squares sides. They are neither pairwise- nor self-intersecting (have no common points). Their ends are on the big square boarder, and all the other vertices are in the big square interior.
Prove that there exists (in addition to four big square angles) a node (corresponding to the cross-lining family, inside the big square or on its side) that does not belong to any broken line.

248 Given natural numbers $x_1, x_2, ..., x_n, y_1, y_2, ..., y_m$.
The following condition is valid: $(x_1 + x_2 + ... + x_n) = (y_1 + y_2 + ... + y_m) < mn$. (*)

Prove that it is possible to delete some terms from (*) (not all and at least one) and to obtain another valid condition.

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249 Given 1000 squares on the plane with their sides parallel to the coordinate axes. Let M be the set of those squares centres. Prove that you can mark some squares in such a way, that every point of M will be contained not less than in one and not more than in four marked squares

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250 Given scales and a set of n different weights. We take weights in turn and add them on one of the scales sides. Let us denote "L" the scales state with the left side down, and "R" – with the right side down.
a) Prove that you can arrange the weights in such an order, that we shall obtain the sequence $LRLRLRLR...$ of the scales states. (That means that the state of the scales will be changed after putting every new weight.)
b) Prove that for every n-letter word containing R's and L's only you can arrange the weights in such an order, that the sequence of the scales states will be described by that word.

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251 Let us consider one variable polynomials with the senior coefficient equal to one. We shall say that two polynomials $P(x)$ and $Q(x)$ commute, if $P(Q(x)) = Q(P(x))$ (i.e. we obtain the same polynomial, having collected the similar terms).
a) For every a find all Q such that the Q degree is not greater than three, and Q commutes with $(x^2 - a)$.
b) Let P be a square polynomial, and k is a natural number. Prove that there is not more than one commuting with P k-degree polynomial.
c) Find the 4-degree and 8-degree polynomials commuting with the given square polynomial P.
d) R and Q commute with the same square polynomial P. Prove that Q and R commute.
e) Prove that there exists a sequence $P_2, P_3, ..., P_n, ...$ (P_k is k-degree polynomial), such that $P_2(x) = x^2 - 2$, and all the polynomials in this infinite sequence pairwise commute.
