All Soviet Union Mathematical Olympiad 1978

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252 Let a_n be the closest to $\sqrt{n}$ integer.
Find the sum $1/a_1 + 1/a_2 + ... + 1/a_{1980}$.

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253 Given a quadrangle $ABCD$ and a point M inside it such that $ABMD$ is a parallelogram. the angle CBM equals to CDM. Prove that the angle ACD equals to BCM.

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254 Prove that there is no m such that ($1978^m - 1$) is divisible by ($1000^m - 1$).

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255 Given a finite set K_0 of points (in the plane or space).
The sequence of sets $K_1, K_2, ..., K_n, ...$ is build according to the rule: [i]we take all the points of K_i, add all the symmetric points with respect to all its points, and, thus obtain K_{i+1}.[/i]
a) Let K_0 consist of two points A and B with the distance 1 unit between them. For what n the set K_n contains the point that is 1000 units far from A?
b) Let K_0 consist of three points that are the vertices of the equilateral triangle with the unit square. Find the area of minimal convex polygon containing $K_n.K_0$ below is the set of the unit volume tetrahedron vertices.
c) How many faces contain the minimal convex polyhedron containing K_1?
d) What is the volume of the above mentioned polyhedron?
e) What is the volume of the minimal convex polyhedron containing K_n?

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256 Given two heaps of checkers. the bigger contains m checkers, the smaller – n ($m > n$). Two players are taking checkers in turn from the arbitrary heap. The players are allowed to take from the heap a number of checkers (not zero) divisible by the number of checkers in another heap. The player that takes the last checker in any heap wins.
a) Prove that if $m > 2n$, than the first can always win.
b) Find all x such that if $m > xn$, than the first can always win.

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257 Prove that there exists such an infinite sequence $\{x_i\}$, that
for all m and all k ($m \neq k$) holds the inequality $|x_m - x_k| > 1/|m - k|$.

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258 Let $f(x) = x^2 + x + 1$.
Prove that for every natural $m > 1$ the numbers $m, f(m), f(f(m)), ...$ are relatively prime.

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259 Prove that there exists such a number A that you can inscribe 1978 different size squares in the plot of the function $y = Asin(x)$. (The square is inscribed if all its vertices belong to the plot.)

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260 Given three automates that deal with the cards with the pairs of natural numbers. The first, having got the card with (a, b), produces new card with $(a+1, b+1)$, the second, having got the card with (a, b), produces new card with $(a/2, b/2)$, if both a and b are even and nothing in the opposite case; the third, having got the pair of cards with (a, b) and (b, c) produces new card with (a, c). All the automates return the initial cards also. Suppose there was $(5, 19)$ card initially. Is it possible to obtain
a) $(1, 50)$?
b) $(1, 100)$?
c) Suppose there was (a, b) card initially $(a < b)$. We want to obtain $(1, n)$ card. For what n is it possible?

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261 Given a circle with radius R and inscribed n-gon with area S. We mark one point on every side of the given polygon. Prove that the perimeter of the polygon with the vertices in the marked points is not less than $2S/R$.

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262 The checker is standing on the corner field of a $n \times n$ chess-board. Each of two players moves it in turn to the neighbour (i.e. that has the common side) field. It is forbidden to move to the field, the checker has already visited. That who cannot make a move losts.
a) Prove that for even n the first can always win, and if n is odd, than the second can always win.
b) Who wins if the checker stands initially on the neighbour to the corner field?

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263 Given n nonintersecting segments in the plane. Not a pair of those belong to the same straight line. We want to add several segments, connecting the ends of given ones, to obtain one non-selfintersecting broken line. Is it always possible?

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264 Given $0 < a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b$.
Prove that $(x_1 + x_2 + ... + x_n)(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}) \leq \frac{(a+b)^2}{4ab}n^2$

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265 Given a simple number $p > 3$.
Consider the set M of the pairs (x, y) with the integer coordinates in the plane such that $0 \leq x < p, 0 \leq y < p$.
Prove that it is possible to mark p points of M such that not a triple of marked points will belong to one line and there will be no parallelogram with the vertices in the marked points.

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266 Prove that for every tetrahedron there exist two planes such that the projection areas on those planes relation is not less than $\sqrt{2}$.

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267 Given $a_1, a_2, ..., a_n$. Define $b_k = \frac{a_1+a_2+...+a_k}{k}$ for $1 \leq k \leq n$.
Let $C = (a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2$, $D = (a_1 - b_n)^2 + (a_2 - b_n)^2 + ... + (a_n - b_n)^2$.
Prove that $C \leq D \leq 2C$.

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268 Consider a sequence $x_n = (1+\sqrt{2}+\sqrt{3})^n$.
Each member can be represented as $x_n = qn + r_n\sqrt{2} + s_n\sqrt{3} + t_n\sqrt{6}$, where q_n, r_n, s_n, t_n are integers.
Find the limits of the fractions $r_n/q_n, s_n/q_n, t_n/q_n$.